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(6) MATHEMATICAL MODELS OF ANTISUBMARINE TACTICS,

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A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

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DOCTOR OF PHILOSOPHY

Field of Industrial Engineering and Management Sciences

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ROGER GLENN SCHROEDER

Evanston, Illinois

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U. S. NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA 93940

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Gentlemen:

Enclosed is a copy of my Ph.D. dissertation, "Mathematical Models of Antisubmarine Tactics." It is felt that the material will be of interest to your readers.

Sincerely,

Roger G. Schroeder
(sk)

ROGER G. SCHROEDER
LTJG, US Navy

Enclosure

SUMMARY

The purpose of this dissertation is to develop mathematical models and solution techniques to find optimal tactics for antisubmarine warfare (ASW) operations. Specifically, two types of ASW operations are considered: (1) a hunter-killer force (P1) is searching for a submarine (P2), and (2) P1 is attacking P2. Both of these types of operations are formulated as two-person zero-sum games. These game formulations distinguish this work from the literature since they allow P2 as well as P1 to choose tactics.

Both sequential and non-sequential search games are developed. For one of the non-sequential games, the search region is divided into n cells. In each play of the game, P1 chooses a cell to search and P2 chooses a cell in which to hide. The resulting payoff is the probability that P1 detects P2. We assume that P1 attempts to maximize this probability of detection while P2 attempts to minimize it. Therefore, the game is zero-sum; and furthermore, P2 is thereby given the role of an evader. We also introduce another similar search game, and we show how to include secondary objectives and additional information ^{are included} by extending these games to constrained game formulations.

Sequential games ~~(where a play consists of several moves)~~ are also developed. When the players move, they not only determine a payoff but also the probability that the play terminates. For the case of at most a finite number of moves, optimal strategies are found by solving a recursive sequence of two-person zero-sum games. For the infinite-move game, we develop an iterative method to approximate the solution to within desired accuracy. Finally, we show that the strategies which minimax the expected duration of the game must also maximin the one-step termination probability.)

↙ To study attack operations, ~~we formulate a model which is a~~ stochastic game due to Shapley. ^{used} In this formulation, a pure strategy is a tactical plan of action for each possible state of the operation. The objective is taken to be either ~~minimax~~ the time or the probability for P1 to kill P2. We derive two methods to find the solution to this stochastic game; one method iterates on the strategies, and the other iterates on the payoffs. One special case which is studied is a Markovian decision process, and one extension is a constrained stochastic game.

↘ ^{are investigated} Finally, ~~we investigate~~ multiple contact problems. Our models rely on the assumption that the amount of effort which is required to accomplish a specified mission is a random variable with a known

distribution function. Several objective functions are employed, and one of the models is a chance-constrained distribution model. By using a zero-order decision rule, we show that the deterministic equivalent of this model is a distribution model with integer extreme points; and hence, optimal integer assignments can be obtained with ordinary non-integer methods.

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CHAPTER I - INTRODUCTION

1.1 Problem Setting

In this chapter we delineate the problems studied, survey the literature, summarize our results, and introduce the elements of game theory and mathematical programming which are used throughout this work. This section defines the type of antisubmarine warfare problems which will be considered. First, we distinguish between strategy and tactics. Then we examine the tactical environment of a hunter-killer force and the interaction between it and a submarine. We close this section by defining the two types of problems which will be studied.

Tactics and strategy differ in the level of decision making, with the lowest level of strategic decisions merging into the highest level of tactical decisions. Furthermore, strategic plans are implemented by tactical operations. In this way, strategy fixes the environment where tactical operations will take place. For example, geographic position, force size, military hardware, and the military mission are usually fixed tactical factors. Tactics then determine the way in which the available forces will be used to achieve given military objectives. See Eccles [1] for further elaboration on these points.

We study tactical situations which are typically encountered by a hunter-killer force in antisubmarine warfare (ASW). First, the primary mission of a hunter-killer force is to seek and destroy submarines.¹ Second, certain environmental factors in ASW operations are fixed. Typical fixed factors for the hunter-killer force (P1) are detection capability, speed, endurance, operating region, and vulnerability. On the other hand, the submarines (P2) also have a mission; and in many cases, it is desirable for P2 to avoid detection in order to accomplish its mission. Hence, in many hunter-killer operations P1 and P2 are in direct conflict, with P1 attempting to detect P2 and with P2 attempting to avoid detection. Our models will deal with these situations of direct conflict, although in some cases we will also allow secondary objectives.

We separate the tactical problems confronting P1 into search problems and contact problems. The essential difference between these problems is the amount of information which is available to P1. In search problems P1 has not established a contact with P2, and in contact situations P1 has a contact. For search problems we will develop models to determine an optimal distribution of search effort, and for contact problems we develop models to find an optimal tactical

¹Sternhell and Thorndike, [1].

configuration of the forces for each possible state of information.

Let us restrict our attention for a moment to search situations. Typically, P1 and P2 each have two modes of search, active and passive. In the active mode, detection devices are emitting electromagnetic radiation and receiving echoes back. The passive mode is simply a listening mode; no radiation is emitted. If P1 operates in the active mode and P2 operates in the passive mode, then P2 can detect P1's signal when the range is too great for the echo to return to P1.

Thus, a passive submarine can detect an active searcher without divulging its own location. It follows that a submarine may choose evasive tactics before a contact is established by an active searcher. Most of the proposed search models in the literature do not allow for an active submarine, but we make this allowance by utilizing game theoretic formulations.

Summing up, we model tactical hunter-killer operations and we separate these operations into search and contact situations. For search situations, the search region and detection capabilities are fixed. Under the assumption that the primary objective of P1 is to detect P2 and that P2 attempts to avoid detection, we determine an optimal distribution of search effort. In contact situations, we wish to find an optimal tactical plan for each state of information. Before outlining our models of these situations and subsequent results, we survey the unclassified literature.

1.2 Literature

No papers on contact problems have been found. However, a large literature on search problems is available. Three important and early papers on search, which encouraged further work, are: Koopman [1] (1946), [2], [3], and [4]; Bellman [1] (1957); and von Neumann [1] (1953). Koopman formulated the first published non-sequential problem on the optimal distribution of search effort; Bellman formulated the first published sequential search model; and von Neumann formulated two non-sequential minimax search problems.

The classical work of Koopman [1] and [4] can be stated as follows: Find a function ϕ which maximizes

$$F(\phi) = \int_R p(x) g(\phi(x)) dx$$

$$(1) \quad \text{Subject to:} \quad \int_R \phi(x) dx = A, \quad \phi(x) \geq 0, \quad x \in R$$

where ϕ is the search density function and $F(\phi)$ is the probability of detection, expressed as an integral of the known submarine probability density $p(x)$ and the conditional probability of detection $g(\phi(x))$.

The constraints require that the total amount of search effort be equal to A and $\phi(x)$ is non-negative over the search region R . Koopman developed a graphical method to solve (1) with the exponential detection function $g(\phi(x)) = 1 - e^{-\phi(x)}$. In 1958, Charnes and Cooper [7] developed a method to obtain an analytical solution to a discrete form of

(1), again with the exponential detection function. Later, de Guenin [1] (1961) obtained an analytic solution to (1) with essentially the requirement that $g(t)$ has a decreasing derivative with increasing t (the detection function exhibits a saturation effect as effort increases). Finally, Zaïni [1] (1963) solved (1) with only a continuity restriction on g . Some results on a sequential version of (1) were obtained by Dobbie [1]. He derived conditions for the optimal distribution of effort $E_1 + E_2$ to be the sum of the optimal distribution of E_1 and the conditionally optimal distribution of E_2 given the submarine has not been found with the effort E_1 .

The following non-sequential search problem was formulated and solved by von Neumann [1] (1953). The search region is divided into n -cells. If P_1 (searcher) and P_2 (hider) both choose the same cell, then P_1 detects P_2 with positive probability; otherwise, the detection probability is zero. P_1 attempts to maximize the probability of detection, while P_2 minimizes it. Hence, the theory of two-person zero-sum games applies. von Nuemann went on to formulate another zero-sum search game which is equivalent to the assignment problem. We discuss both of these problems in further detail in Chapter 2. Neuts [1] (1963) extended von Neumann's n -cell search game to an infinite number of moves, but only a certain type of detection function is permitted. We also discuss this extension in Chapter 2. This completes

our discussion of basic non-sequential search models and extensions to the sequential case.

In 1957, Bellman [1] formulated an n -cell sequential search problem. He assumed that the searcher has a prior probability distribution on the location of the submarine and that the cost of searching a cell is also known. The searcher then looks in one cell at a time until the submarine is found. Bellman found the policy (sequence of cells to search) which minimizes the total expected cost. Gluss [1] (1961) added a search cost to Bellman's problem which depends on the distance between successive looks (moving cost). He found an optimal policy for several important cases of the prior probabilities.

Various sequential search models have been formulated where the prior probabilities are transformed to Bayesian posterior probabilities. Neuts [1] extended Bellman's model to include Bayesian's updating of prior probabilities. Norris [1] employs a minimax optimization for the Neuts model, but his results were only complete for the case of two cells. Finally, Pollock [1] formulated a Bayesian test of hypothesis model. At the beginning of this sequential search, the searcher estimates the probability p that a submarine is present in the search region. Each time a search is made, p changes according to the Bayesian rule. After each search is made, the searcher takes one of the following three decisions: (1) make another search, (2) accept the

hypothesis H_0 that a submarine is present, or (3) reject H_0 . The optimal decision depends on the current estimate of p and the costs of wrong decisions.

The last type of search models which we discuss will only be mentioned in passing. These are n -cell search models where a submarine arrives in some cell of the search region at a random time during the search. Such models have been studied by Blackman [1], Blackman and Proschan [1], and Pollock [1].

1.3 Results

Bringing together the discussion of the last two sections, it becomes apparent that models are required for hunter-killer operations which allow the submarine as well as the searcher to make tactical decisions. Most authors assume that the submarine maintains a known stationary probability distribution. We do not make this assumption and furthermore we allow the probability of detection to be a function of range. All of the distribution of effort models reviewed in the last section have assumed a negligible radius of detection, and all of the models, except the minimax models, assume a stationary submarine. With these observations in mind, we preview the models and results obtained in the next four chapters.

We introduce our results by chapter. The first model of Chapter 2 is an n -cell search game. This game is non-sequential but extensions of it to a sequential game are made in Chapter 3. We assume that P1 attempts to maximize the probability of detection while P2 attempts to minimize it. In reality there doesn't have to be a submarine present in the searching region for this model to apply and P2 may consist of one or more submarines. But, we are assuming that P1 should act as if an evading submarine was present. Hence, we seek a distribution of effort for P1 which will maximize the minimum probability of detection against all possible hiding strategies that P2 can choose. Our model

is also formulated to allow the probability of detection to be a function of range.

Now the above model only applies when neither P1 nor P2 have information on where the other player is searching or hiding. However, P1 may have intelligence information which can be used to bound the probability that P2 is located in certain cells or these bounds may arise from previous searches. If such information is available, then the optimal distribution of effort obtained from the foregoing model will be too "conservative". To take into account certain information on hiding locations we show how to extend the game to a constrained game.¹

This extension will make P1's strategy less conservative but perhaps more risky. We also show how to include other types of information which may arise in searching situations. Finally, we give an example of this model and obtain an analytic solution for the special case of a negligible radius of detection.

We also propose a second model in Chapter 2. This model is especially suited to search in sweeps of the search region. For example, search by aircraft. Again the opposing objectives of detection and evasion are assumed, and the searcher seeks to minimax the probability of detection. The game formulation of this problem is reduced to

¹The notion of a constrained game as incorporating "habits" or other qualitative probabilistic information was introduced by Charnes [1], 1953.

a dyadic model, a generalization of the distribution problem. Special cases of this game result in a transportation and an assignment problem. The chapter is completed with tactical examples of this model. Both of these games are generalization of von Neumann's [1] search games.

In Chapter 3, we develop a sequential extension of the n -cell search model of Chapter 2. When the players each choose a cell, they not only determine a probability of detection but they also determine a probability that the game is played again. We consider both a finite and an infinite number of moves. In the finite case, optimal strategies can be found by solving a recursive sequence of two-person zero-sum games. Substantially less computational effort is required by this procedure than solution of the normalized form of this game. For the infinite game, the problem is reduced to finding strategies X and Y which minimax the form $\frac{X^t A Y}{X^t Q Y}$, where A is the payoff matrix for each move and Q is the matrix of non-zero stop probabilities. We show how to find the optimal strategies by solving a linear programming problem with a parameter in the constraint set. We demonstrate that optimal strategies are obtained when this parameter is chosen to make the optimal value of the objective function equal to zero. Then we develop a technique to find such a value of the parameter in a finite number of steps. Chapter 3 is concluded with an example which compares the

non-sequential game to both the finite and infinite sequential search games.

In Chapter 4 we study the contact problem. However, the models which are developed may also be applied to some types of search situations. Briefly, we view the contact problem as a certain game of pursuit between P1 and P2. This game consists of a finite collection of states and each state corresponds to a possible tactical configuration of the hunter-killer forces. At each move, the players observe the state of the game and each player chooses a tactical plan from a finite collection. The observed state and the chosen tactical plans jointly determine an immediate payoff and a transition probability distribution over the states. Before the next move is made, the game transits to one of the states or terminates according to the chosen probability distribution. We seek to find an optimal strategy for each player. An optimal strategy is one of a minimax pair for the total expected payoff.

This game is a stochastic game due to Shapley [1]. He defined a vector value and employed an ingenious argument to establish its existence and that of optimal strategies. In addition, he showed that the value and optimal strategies are characterized by a non-linear fixed point problem. We show how to approximate the solution to this fixed point problem by linear programming methods. Two methods are given;

one iterates on the strategies and the other iterates on the payoffs.

To introduce more realism into the game, several variants of the stochastic game are considered. One of these variants is the extension of Charnes' notion of a constrained game to stochastic games. We also examine a stochastic game with perfect information which is a terminating Markovian decision process and we extend some of the known results. Finally, we introduce a finite version of the stochastic game and show how our linear programming methods may be used to obtain a solution. All of the above models are examined in light of their tactical consequences and their applications to hunter-killer operations.

We turn to a different type of problem in Chapter 5 than those considered thus far. Here we are concerned with distributing hunter-killer forces to multiple contacts. Four models are formulated starting with simple situations and progressing to a dynamic problem. The first two models deal with the allocation of a fixed number of units to two or more contact areas. A specified mission is to be accomplished in each area but the number of units required for this purpose is a random variable. This random requirement may arise due to insufficient intelligence on enemy capabilities and objectives or other uncertainties.

In the first model, we introduce a novel objective function.

The objective is to maximize the probability that all requirements are met or equivalently maximize the probability that all missions are simultaneously accomplished. The constraint set consists of a single constraint on the total amount of effort available and non-negativity restrictions. We develop an algorithm to find the analytic solution when the requirements are uniformly and independently distributed. The second model is obtained by taking the following objective: minimize the sum of the expected shortages. The resulting model is a problem in generalized constrained medians as discussed by Charnes, Cooper, and Thompson [2]. We find that the assumption of uniformly and independently distributed requirements, in this case, leads to a quadratic programming problem.

Next, we examine tactical situations where the distribution time is an important measure of effectiveness. Here we obtain a chance-constrained distribution (transportation) model. The availabilities are known but again the requirements for a specified mission in each contact area are discrete random variables. The deterministic equivalent for this problem has discrete availabilities and requirements and, therefore, non-integer distribution techniques may be employed to obtain an optimal integer solution.

The final model is simply a dynamic two-period version of the above distribution model. Here, we employ a zero-order decision rule

for both periods; and a method is given to allow the second period allocations to depend on the requirements observed in the first period.

In addition, we indicate how a linear-decision rule can be applied to this type of model.

1.4 A Survey of Mathematical Programming

Relevant topics to this work in mathematical programming are surveyed. This survey is intended for the well-versed reader and only an orientation to several important topics is desired. Therefore, the treatment is brief and no extensive literature citations are included. We cover the following topics: the transportation problem; linear programming, including duality; and some aspects of chance-constrained programming. Only the models and the main theorems are presented. The reader is referred to the literature for a discussion of standard solution techniques such as the simplex method.

The modern form of the transportation problem was first formulated and studied by Hitchcock [1], although even more general forms of this problem were studied as early as 1939 (Kantorovich), but were not available until some years after World War II.¹ The transportation model may be visualized by supposing that there are m shipping points (origins) with a_i units available at origin i ($i = 1, \dots, m$) and n destination with b_j units required at destination j ($j = 1, \dots, n$). Units can be shipped from each origin to any destination and a shipping cost of c_{ij} is incurred when one unit is shipped from origin i to destination j . The problem is to find a shipping schedule (number of units to be sent

¹See Charnes and Cooper [5] for an extensive discussion of historical developments and early work.

from each origin to each destination) which minimizes total cost. Accordingly, we let x_{ij} be the number of units to be shipped from origin i to destination j . Then the mathematical problem (model) may be stated as follows. Find the values of x_{ij} which¹

$$\begin{aligned} & \text{Min } \sum_{i,j} c_{ij} x_{ij} \\ \text{Subject to: } & (2a) \sum_j x_{ij} = a_i \\ & (2b) \sum_i x_{ij} = b_j \\ & x_{ij} \geq 0 \end{aligned}$$

Constraints (2a) and (2b) require that the total amount sent from each origin is equal to the amount available there and the total amount sent to each destination is equal to the requirement.

The following well-known² properties of (2) are immediately displayed.

(a) Problem (2) has an optimal solution if and only if

$$\sum_i a_i = \sum_j b_j$$

(b) If the a_i and b_j are all integers, then every basic feasible solution to (2) has integer-valued variables.³

¹The subscripts vary over their entire range when the range is not indicated.

²See Charnes and Cooper [5] for further elaboration.

³A basic feasible solution has no more than $m+n-1$ (the number of linearly independent constraints) positive variables.

It follows that at least one optimal solution has integer valued variables and the optimal integer solutions may be found by the usual non-integer adjacent extreme point methods.

The above properties are utilized in Chapters 2 and 5. We also encounter inequalities in the constraints (2a) and (2b), but we show how to reduce these inequality forms to the standard form when this reduction is needed.

The most widely used method to solve (2) consists of three steps.

- (a) Find a basic feasible solution (b. f. s.).
- (b) Evaluate the current b. f. s. for optimality.
- (c) If the b. f. s. is not optimal, move to another b. f. s. which decreases the value of the objective function and return to step (b).

If (2) has a feasible solution, then the above method converges to an optimal solution in a finite number of steps. In terms of geometry, this method is an adjacent extreme point method; and the key mathematical property which makes the method work is the equivalence of basic feasible solutions and extreme points of the convex set of feasible solutions. A more detailed discussion of these ideas would lead us too far afield from the purpose of this survey. Therefore, we turn to a brief discussion of linear programming.

Next, we define a linear program and then give the dual theorem of linear programming. A linear program consists of a linear objective function which is to be optimized and linear constraints. In addition, non-negativity restrictions on the variables are usually stated separately. Every linear program may be written in the following form

$$\begin{aligned} \text{Max } c^t X \\ (3) \quad AX \leq b \\ X \geq 0 \end{aligned}$$

where c and X are n -vectors, b is an m -vector, A is an $m \times n$ matrix, and c , b , and A all have constant elements. The set $S = \{X \mid AX \leq b, X \geq 0\}$ is called the set of feasible solutions. The problem is to find an $\hat{X} \in S$ which maximizes $c^t X$ over all $X \in S$. If such an \hat{X} exists and is finite, then we call \hat{X} an optimal solution to (3). Not every linear program has an optimal solution but exactly one of the following three cases must occur:

- (a) No feasible solution exists.
- (b) An optimal solution exists.
- (c) $c^t X$ is unbounded for some $X \in S$.

As in the transportation problem, the methods which are available to solve a linear program depend on the equivalence of extreme points and basic feasible solutions, and on the optimality of at least one basic feasible solution, if an optimal solution exists. Of these adjacent

extreme point methods, the most widely used methods are the simplex method of Dantzig [2] and the dual method of Lemke [1]. The simplex method consists of the same steps as those outlined for the transportation problem; however, the means of going from one step to the next are different in each case. It starts from a basic feasible solution (b. f. s.) and several techniques are available to get an initial b. f. s. With a starting technique and the simplex method, one will arrive at an optimal solution or case (a) or (c) above in a finite number of steps.

With every linear program there is associated another linear program called the dual. The dual to problem (3) is defined as

$$\begin{aligned}
 &\text{Min } W^t b \\
 (4) \quad &W^t A \geq c^t \\
 &W \geq 0
 \end{aligned}$$

There are certain surprising relations between the solutions to (3) and its dual (4). These relationships are summed up by the

dual theorem: Solutions to the primal (3) and dual (4) are related as follows:

- (i) Problem (3) has an optimal solution if and only if (4) has an optimal solution.
- (ii) When (3) and (4) have optimal solutions \hat{X} and \hat{W} respectively, then $c^t \hat{X} = \hat{W}^t b$.
- (iii) If either (3) or (4) has an unbounded solution, then the other problem has no feasible solution.

Proof - see Charnes and Cooper [5] for details.

In addition, it is possible for both problems to have no feasible solution.

Therefore, if a linear program has no feasible solutions, its dual is either unbounded or infeasible.

The practical significance of duality relationships lie primarily in the sensitivity information which is available. It can be shown that \hat{W}_i is the change in the primal objective function per unit change in b_i . But, the physical interpretation of \hat{W}_i will depend on the actual physical process which is modeled. Another important point which should be made here is that an optimal solution to the dual problem is available when the primal is solved by the simplex method. Hence, sensitivity information is immediately available.

Next, we give the theorem of the alternative¹ which follows immediately from the dual theorem.

Theorem of the alternative: Suppose \hat{X} and \hat{W} are optimal solutions to (3) and (4) respectively. Let \hat{X}_s and \hat{W}_s be the slack vectors for these optimal solutions. Then

$$\hat{W}^t \hat{X}_s = 0, \quad \hat{W}_s^t \hat{X} = 0$$

This result is quite useful for analysis and it also provides additional primal-dual interpretations. We continue now with pertinent topics from chance-constrained programming.

The idea of chance-constrained programming was first introduced by Charnes, Cooper, and Symonds at the December 1953 meeting

¹This theorem is also known as the principle of complementary slackness.

of the Econometric Society. But, four years of refereeing elapsed until it was published in journal form in [1] 1958). Since then a number of problems have been treated, but we restrict our discussion to a definition of chance-constrained programming and the so called "zero-order decision rule". This rule will be applied to a transportation-type problem in Chapter 5.

To define chance-constrained programming (ccp), we draw on the following definition given by Charnes & Cooper [4].

"Chance-constrained programming admits random data variations and permits constraint violations up to specified probability limits. Different kinds of decision rules and optimizing objectives may be used so that, under certain conditions, a programming problem (not necessarily linear) can be achieved that is deterministic - in that all random elements have been eliminated."

The reader may refer to Charnes and Cooper [4] and subsequent papers for some general formulations of a ccp. We restrict our attention here to the following problem ("zero-order decision rules"):

$$(5) \quad \begin{aligned} & \text{Max } E(c^t X) \\ (5.1) \quad & \text{Pr} \{AX \leq b\} \geq d \end{aligned}$$

where "E" denotes the expectation operator, c and X are n -vectors, b and d are m -vectors, and A is an $m \times n$ matrix of constants. We assume that b is a vector of random variables with a known continuous joint cumulative distribution function (c. d. f.). We further assume that c is a vector of random variables with known and finite means. The double

inequality in (5.1) reads as follows: The probability that $AX \leq b$ is no less than d . Hence the constraint $AX \leq b$ can be violated but it must be satisfied with at least joint probability d .

We rewrite (5.1) as

$$(6) \quad \Pr (a_i X \leq b_i) \geq d_i \quad i = 1, \dots, m$$

where a_i is the i^{th} row of the matrix A and b_i and d_i are the i^{th} components of b and d respectively. Let F_i be the marginal c. d. f. of the random variable b_i . Then (6) is equivalent to

$$(7) \quad 1 - F_i (a_i X) \geq d_i \quad i = 1, \dots, m$$

provided we are using zero-order decision rules, i. e., X is not a function of the random variables and A is a constant matrix. Since F_i is monotone increasing and continuous, F inverse, F_i^{-1} , exists. It follows immediately that X satisfies (7) if and only if X satisfies

$$a_i X \leq F_i^{-1} (1 - d_i) \quad i = 1, \dots, m$$

Because of this relationship, (5) is equivalent to

$$(8) \quad \begin{aligned} &\text{Max } c^t X \\ &a_i X \leq F_i^{-1} (1 - d_i) \quad i = 1, \dots, m \end{aligned}$$

where c is the vector of mean values of c . The above linear program is called the deterministic equivalent of (5). From the linearity of (8), all of the relationships of linear programming including duality can be brought to (8) or equivalently to (4).

In multiperiod models it is often desirable to determine the optimal value of X adaptively. This is to say, X should depend on the actual values of b which are observed. This dependence gives rise to the notion of a decision rule as defined by Charnes and Cooper [5] and extensively studied in Charnes and Kirby [1]. The latter prove the optimality of piece-wise linear decision rules for the E-model. Charnes and Cooper have particularly studied the class of linear decision rules

$$X = Db + a$$

where the elements of the matrix D and the vector a are unknown constants. These constants are to be determined by reference to (5). The above relationship for X is substituted into (5) and Charnes and Cooper [5] then obtain a deterministic equivalent convex programming problem when the random variables b are normally distributed. Solving this deterministic equivalent yields an optimal D and a which in turn specifies an optimal X for each observed b , via the above linear rule. Additional material on chance-constrained programming may be found in the references listed under Charnes, et. al.

1.5 Elements of Game Theory

We discuss certain elementary concepts from the theory of games which will be used extensively. The following topics are considered in turn: definitions for a game, minimax theorem, linear programming formulation, extensive form, and Kuhn's theorem of perfect recall. The first formulation of the modern theory of games and the minimax theorem was given by von Neumann [2]. Subsequently, von Neumann and Morgenstern [1] brought the theory to a high state of development. Additional contributions are scattered throughout the literature. However, a large number of these contributions are contained in the Princeton series of "Contributions to the Theory of Games" and the Proceedings of the National Academy of Sciences (U. S. A.). In addition, several books on the theory of games are: von Neumann and Morgenstern [1], Blackwell and Girshick [1], Dresher [1], Karlin [1], Luce and Raiffa [1], and McKinsey [1].

In order to provide a common ground for discussion, it is necessary to introduce several definitions.

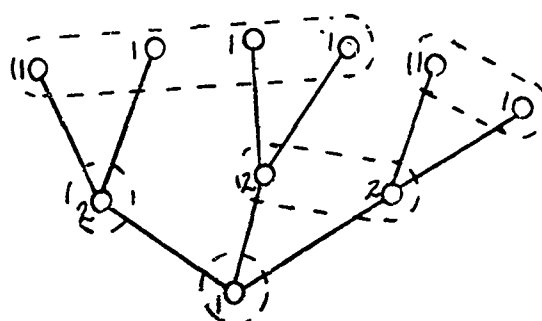
1. A game is defined by the totality of its rules.
2. A play of the game is one complete execution of the set of rules.
3. A move is defined as a point in the game when one of the players must choose an alternative.

4. An alternative is one of the choices which a player may take when it is his move.

The rules of a game distinguish one game from another. They specify the sequence in which the players move, the amount of information which is available to each player, what the payoffs are, how a play terminates, and the alternatives which are available. The rules determine a payoff to each player in the following way: Let M_1, M_2, \dots, M_m represent the sets of alternatives at the moves in a game and let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_m)$ be a sequence of alternatives with $\sigma_i \in M_i$ ($i = 1, \dots, m$). Then the sequence σ is a play of the game. Suppose there are n players denoted by P_1, P_2, \dots, P_n . Now the rules specify a set of functions $\{F_j(\sigma) \mid j = 1, \dots, n\}$ for each play σ with $F_j(\sigma)$ being the payoff received by P_j . If for some σ , $F_j(\sigma) \geq 0$ then P_j receives the amount $F_j(\sigma)$ and if $F_j(\sigma) < 0$ then P_j pays the amount $F_j(\sigma)$. A game is called zero-sum if $\sum_{j=1}^n F_j(\sigma) = 0$ for each σ . In the remainder of this discussion we restrict our attention to two-person ($n = 2$) zero-sum games. These games describe situations of conflict between two opposing interests and they are used exclusively to model the tactical situations under consideration.

von Neumann and Morgenstern [1] introduced the useful notion of the extensive form of a game. In this form, a game is viewed as a tree consisting of nodes and branches. A node corresponds to a move

for a particular player and the branches emanating from a node represent the alternatives which are available. The origin of the tree corresponds to the first move and successive nodes correspond to successive moves. Then each play is represented by a unicursal path from the origin of the tree to a terminal branch and each terminal branch corresponds to precisely one play of the game. We illustrate these ideas with the following example:



Example 1

The number beside each node designates which player's move it is. The dotted lines define information sets. Roughly speaking, an information set tells us what a player knows at each move. A player will know which information set he is in, but not which node he is at. To illustrate the concept of information, we give the following scenario of the above game. P1 moves first and chooses one of three alternatives.¹ The information sets for the next move, P2's move, tell us if P1 chooses

¹For convenience, we number alternatives in a clock-wise direction.

alternative 1; then P2 is informed of this but if P1 chooses alternative 2 or 3, then P2 is not informed of the specific alternative chosen by P1. Now on P1's second move, P1 is forced to forget whether he had chosen alternative 1 or 2 on the first move but P1 remembers whether he had taken alternative 3 or not. Further, P1 does not know which alternative is taken by P2.

We define a pure strategy. A pure strategy for a player is a function from the set of all possible histories of the game into the sets of alternatives. It is a specification of an alternative at each move for each possible history up to the move. In example 1, player 2 has eight pure strategies corresponding to the eight ways to map the set $\{1, 2, 3\}$, into the set $\{1, 2\}$. Each of these ways is a specification of the alternative P2 should choose (1 or 2) depending on what P1 chooses (1, 2, or 3). In a similar way we can enumerate the pure strategies for P1. Here a pure strategy tells P1 what alternative to choose for both his moves as a function of the history of the game.

von Neumann and Morgenstern [1] have shown that every finite two-person zero-sum game can be reduced to normal form. In normal form the game is represented as an $m \times n$ matrix A with each row corresponding to a pure strategy for P1 and each column corresponding to a pure strategy for P2. If P1 chooses row i and P2 chooses column j then P1 receives a_{ij} from P2 and since the game is zero-sum, P2 receives

$-a_{ij}$ from P1. We refer to these games as matrix games, and A is called P1's payoff matrix. With the game in normal form, we now think of it as consisting of one move by each player with the pure strategies being alternatives. Now either player may move first, and the second player to move is in ignorance of the alternative chosen by the first player; or equivalently the players may move simultaneously in ignorance of the other's move.

The normal form is convenient for a discussion of rational play and mixed strategies. von Neumann's [2] concept of rational play requires that each player maximize his minimum expected payoff or simply minimax the expected payoff. To get to the heart of this matter, we introduce mixed strategies and the minimax theorem of von Neumann

[2]. A mixed strategy for a player is a probability distribution over the available alternatives (pure strategies). Accordingly, we assume P1 plays alternative i with probability x_i ($i = 1, \dots, m$) and P2 plays his alternative j with probability y_j ($j = 1, \dots, n$). We call the $m \times 1$ vector $X = (x_1, \dots, x_m)$ and the $n \times 1$ vector $Y = (y_1, \dots, y_n)$ mixed strategies for P1 and P2 respectively. Since X and Y are probability distributions, we must have

$$\sum_{i=1}^m x_i = 1, \quad x_i \geq 0; \quad \sum_{j=1}^n y_j = 1, \quad y_j \geq 0$$

We let $E(X, Y) = X^t A Y$, where A is P1's payoff matrix. Then $E(X, Y)$

is the expected payoff to P1 when P1 chooses X and P2 chooses Y . The following remarkable theorem consolidates the theory of matrix games.

Minimax theorem (von Neumann). For every matrix A , there exists strategies \hat{X} and \hat{Y} such that

$$(9) \quad E(X, \hat{Y}) \leq E(\hat{X}, \hat{Y}) \leq E(\hat{X}, Y) \quad \text{all strategies } X \text{ and } Y.$$

The strategies \hat{X} and \hat{Y} are called optimal strategies for P1 and P2 respectively, and $v = E(\hat{X}, \hat{Y})$ is termed the value of the game.

We can immediately interpret the meaning of an optimal strategy and equivalently rational play. From equation (9), if P1 plays \hat{X} , then he receives at least v regardless of the strategy P2 employs. Furthermore, P2 can prevent P1 from getting more than v by playing \hat{Y} . Hence, a player can gain nothing by deviating from an optimal strategy and he can lose more if he does deviate from an optimal strategy. The minimax theorem settles important questions of the theory but it does not tell us how to compute optimal strategies.

Next, we show how optimal strategies may be computed by linear programming. Consider the following dual pair of linear programs¹

¹This formulation is a variant of the one in Charnes [1].

Max v

Min u

$$v - \sum_{i=1}^m x_i a_{ij} \leq 0; \quad j=1, \dots, n \quad u - \sum_{j=1}^n a_{ij} y_j \geq 0; \quad i=1, \dots, m$$

$$(10) \quad \sum_{i=1}^m x_i = 1$$

$$(11) \quad \sum_{j=1}^n y_j = 1$$

$$x_i \geq 0 \quad i=1, \dots, m \quad y_j \geq 0 \quad j=1, \dots, n$$

In (10) and (11) the matrix $A = (a_{ij})$ is to be interpreted as P1's payoff matrix and P1 is the maximizing player. Charnes [1] has shown that optimal solutions $\hat{X} = (\hat{x}_1, \dots, \hat{x}_m)$, $\hat{Y} = (\hat{y}_1, \dots, \hat{y}_n)$ and $\hat{u} = \hat{v}$ exist for (10) and (11) and that

$$X^t A \hat{Y} \leq \hat{X}^t A \hat{Y} \leq \hat{X}^t A Y \quad \text{all strategies } X \text{ and } Y.$$

Therefore, optimal solutions to (10) and (11) correspond to optimal strategies in the matrix game A. Further, when either (10) or (11) is solved the optimal solutions to the other program are available. Therefore, the value and optimal strategies may be found by solving a single linear program.

To avoid confusion later, we emphasize that P2 does not have to receive the negative of the payoff that P1 receives in order for zero-sum theory to apply. Indeed, we do not postulate negative payments later when the payoff is taken to be a probability or a unit of time. Nevertheless, zero-sum theory applies because one player is attempting to maximize the expected payoff and the other player seeks

to minimize it - this is all that is really necessary.

When games are formulated directly in the matrix form, zero-sum theory requires that each player be in ignorance of the other player's choice. In most real-world situations and particularly in tactical encounters, such total ignorance does not prevail. Some form of intelligence or habits of the opposition are usually known. To incorporate this type of information and other types, we utilize the constrained game formulation of Charnes [1]. This approach is employed in Chapter 2, and we give a complete discussion of it there.

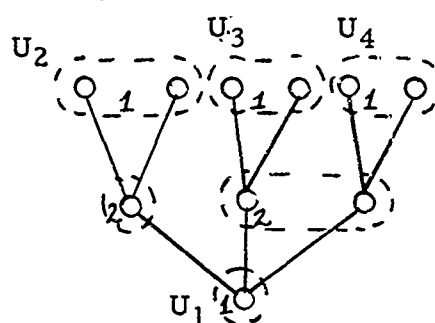
We return to the extensive game form to discuss perfect information and perfect recall. A game has perfect information if each information set contains exactly one node. This means that when each player moves he must know the complete history of the game including the other player's moves. Of course, it is well-known that there exist optimal pure strategies for a game with perfect information.¹ Intuitively speaking, when a game has perfect information, randomizing is not necessary to hide a player's choice since it will be disclosed to the other player in subsequent moves. For example, checkers is a game with perfect information.

The concept of perfect recall was introduced by Kuhn [1], and

¹Refer to von Neumann [2].

it is an extension of the notion of perfect information. The fundamental result is that a behavior strategy is optimal in a game of perfect recall. Vaguely speaking, a game has perfect recall if every player remembers which alternative he took in all preceding moves. However, he does not need to be informed of the alternatives which were chosen by the other players.

To illustrate, we can change example 1 to a game of perfect recall by redefining the information sets for P1 on his second move as follows:



Now on P1's second move he remembers which alternative he has taken on his first move (Of course, there also are other ways to introduce perfect recall into this game.).

To define a behavior strategy, we assume that, say P1, has n information sets and we let X_i be a mixed strategy over the alternatives available in information set i ($i = 1, \dots, n$). Then $X = \{X_1, X_2, \dots, X_n\}$

is a behavior strategy for P1. In the above example, a behavior strategy for P1 is $X = \{X_1, X_2, X_3, X_4\}$ where X_i is a mixed strategy over the information set U_i ($i = 1, 2, 3, 4$). For example, $X_3 = (a, 1-a)$, where a is the probability of choosing alternative 1 in U_3 . Since P2 only has one move, his behavior strategy is the same as his mixed strategy. It is easy to construct games where one can do better with a mixed strategy than with a behavior strategy. It has been shown by Kuhn [1], on the other hand, that a game of perfect recall always has optimal behavior strategies.

This concludes our brief discussion of game theory and our introductory chapter. We turn to the development of models and methods for ASW tactics.

CHAPTER II - NON-SEQUENTIAL SEARCH GAMES

2.1 Introduction

Two deterministic search games are developed in this chapter. These games are idealizations of tactical situations which arise in anti-submarine warfare. We study searching problems where a hunter-killer force, P1 (player 1), and a submarine, P2 (player 2), are in direct conflict. Specifically, we consider tactical problems where P1 attempts to detect P2 and P2 attempts to avoid detection. Because of the opposing military objectives of detection and evasion, these tactical problems may be formulated as two-person zero-sum games. An appropriate payoff function is defined to reflect the objectives of detection and evasion, and we show how optimal strategies correspond to optimal deployment plans.

We also consider constrained game extensions of the basic search games and thereby allow secondary military objectives in addition to the primary objectives of detection and evasion. In addition, these constrained games permit the players to choose optimal strategies based on intelligence or information on the opposing players' tactics derived, perhaps, from previous attacks. Hence, optimal strategies employed in a particular play may depend on actual information obtained from

previous encounters.

Examples are given of particular tactical situations which are encompassed by the games, and these examples serve to illustrate additional features of the models. Special cases are also treated; and in section 2.6, we obtain an analytic solution for a special case of the first game. Both of the games are shown to be generalizations of search games introduced by von Neumann [1].

2.2 Formulation of the n-Cell Game

The n-cell search game is played within a specified search region which is apportioned into n cells numbered $i = 1, \dots, n$. A pure strategy for P1 is a cell to search, and a pure strategy for P2 is a cell in which to hide. Hence, each player has n pure strategies; one corresponding to each cell. A play of the game consists of a simultaneous choice of strategies by the players. Of course, the same game obtains if the players choose their strategies successively provided that the second choice is made in ignorance of the first.

Now we define an objective and an appropriate payoff function. The primary mission of a hunter-killer force is to seek and destroy submarines, Sternhell and Thorndike [1]. We deal here with the seeking aspect of hunter-killer operations, and focus our attention on tactical situations where P1 attempts to detect P2 while P2 attempts to avoid detection. Hence, a reasonable measure of effectiveness for each pair of fixed strategies is the probability that P1 detects P2. To formulate this measure of effectiveness, we postulate the following payoffs. Let p_{ij} ($i, j = 1, \dots, n$) be the conditional probability that P1 detects P2 given P1 searches cell i and P2 hides in cell j; and let P be the $n \times n$ matrix $P = (p_{ij})$. Let the $n \times 1$ vectors $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ be mixed strategies for P1 and P2 respectively. Now x_i is the probability that P1 searches cell i and y_j is the probability that

that P2 hides in cell j . Thus, the probability that F detects P2 for the mixed strategies X and Y is simply

$$(1) \quad X^t P Y$$

Equation (1) is the desired measure of effectiveness for the type of hunter-killer operations under consideration. We assume that P1 is the maximizing player and P2 the minimizing player.

The celebrated minimax theorem of von Neumann [2] establishes the existence of strategies \hat{X} and \hat{Y} and a real number \hat{v} (the value of the game) which satisfy the equation

$$(2) \quad X^t P \hat{Y} \leq \hat{v} = \hat{X}^t P \hat{Y} \leq \hat{X}^t P Y \quad \text{all strategies } X \text{ and } Y$$

or we may also write

$$(3) \quad \hat{v} = \max_X \min_Y X^t P Y = \min_Y \max_X X^t P Y = \hat{X}^t P \hat{Y}$$

From equation (2), if P1 plays an optimal strategy \hat{X} , then the total payoff (probability of detection) is at least as great as \hat{v} regardless of P2's strategy. Similarly, if P2 plays an optimal strategy \hat{Y} , the total payoff is no greater than \hat{v} regardless of P1's strategy. It follows that P1 can choose a strategy to maximize the probability of detection while P2 simultaneously minimizes it. Hence, the conflicting objectives of detection and evasion are embodied in the given two-person zero-sum game formulation.

An important feature of the n -cell search game is that P1 may

detect P2 with positive probability from anywhere in the search region. This feature permits us to consider tactical situations in which the probability of detection is a function of the range between P1 and P2. Variation of detection probability with range is a basic property of detection devices,¹ but none of the search models referenced in the bibliography permit this variation. They assume either explicitly or implicitly that the radius of detection is negligible. This assumption for the n-cell game requires P to be a diagonal matrix, and we take up this special case in section 2.6.

¹See Morse and Kimball [1].

2.3 Solution and Tactical Interpretation

We present a method to compute the solution to the n-cell search game. No computational advantage is gained from the fact that the payoff elements are probabilities. Hence, we employ a computational method developed for a general two-person zero-sum game.

Optimal strategies and the value may be computed and are characterized by the following dual linear programs due to Charnes [1].

Max v

Min u

$$v - \sum_{i=1}^n x_i p_{ij} \leq 0; j=1, \dots, n \quad u - \sum_{j=1}^n p_{ij} y_j \geq 0; i=1, \dots, n$$

$$(4) \quad \sum_{i=1}^n x_i = 1$$

$$(5) \quad \sum_{j=1}^n y_j = 1$$

$$x_i \geq 0 \quad i=1, \dots, n \quad y_j \geq 0 \quad j=1, \dots, n$$

Let $\hat{X} = (\hat{x}_1, \dots, \hat{x}_n)$, $\hat{Y} = (\hat{y}_1, \dots, \hat{y}_n)$, and $\hat{u} = \hat{v}$ be an optimal solution to problems (4) and (5). Then from Charnes [1]

$$\hat{X}^t P \hat{Y} \leq \hat{v} = \hat{X}^t P \hat{Y} \leq \hat{X}^t P Y \quad \text{all strategies } X \text{ and } Y.$$

Hence, \hat{X} and \hat{Y} are optimal strategies and \hat{v} is the value of the game, see equation (2)).

We bring together the following assumptions which have been made and examine them in light of their tactical consequences.

1. Both players know the game is being played.
2. Both players are given the search region and the particular subdivision of it into cells, i. e., they know what pure strategies are available.

3. Both players are given the payoff matrix.
4. P1 acts to maximize the probability of detecting P2 and P2 minimizes this probability.

We study the above assumptions from P1's point of view when P1 has all of the information necessary to satisfy the above assumptions and P2 has part or perhaps none of the information. Hence, suppose that P1 goes out and specifies a search region of interest and divides it into cells. Now P1 will know the detection characteristics of his own searching equipment and can therefore construct a payoff matrix. Further, assume that P1 wishes to minimax the probability of detecting P2. Then the above assumptions are satisfied for P1. Now, it is unlikely that P2 will also have all of the information required by assumptions 1 through 4. Thus, due to ignorance of the essentials of the game, P2 may not play an optimal strategy. Nevertheless, if P1 plays an optimal strategy, then P2 is detected with probability at least as great as the value of the game. For practical purposes, it is therefore immaterial whether P2 has all of the information required by assumptions 1 through 4. The important point is that when P1 plays an optimal strategy he is acting as if P2 does have all of the required information and P2 may, in fact, have a substantial amount of it.

Consider the situation where P1 employs an optimal strategy for several plays of the game and his strategy is discovered by P2. Then from equation (2), P2 cannot take advantage of the fact that he has

discovered P1's optimal strategy because P1 will detect P2 with probability at least as great as the value of the game regardless of P2's strategy. Therefore, P1 can employ the same optimal strategy throughout several plays of the game without risking adverse consequences.

Finally, we give an interpretation of optimal strategies for P1 in terms of optimal search plans. A search plan, in the sense used here, is a specified configuration of the available search effort or equivalently the amount of effort which is to be assigned to each cell. If P1 has a single unit of effort which is indivisible, then he may play the optimal strategy \hat{X} by searching cell i ($i = 1, \dots, n$) with relative frequency \hat{x}_i . Of course, these relative frequencies may be realized over several plays of the game by selecting a pure strategy for each play at random from the distribution \hat{X} . Now suppose that P1 has a total amount of effort E available which is infinitely divisible. For example, E may be the number of flying hours available for searching which is approximately infinitely divisible. In this case, P1 may allocate the amount of effort $\hat{x}_i E$ to cell i ($i = 1, \dots, n$), and this allocation is optimal with respect to the game model. Hence, an optimal search strategy may correspond to an optimal search plan.

2.4 Tactical Examples

We give a tactical example of the n-cell game model. Suppose that submarines (P2) must pass through a channel to get from their bases to operating areas. P1 wishes to set up a patrol barrier across the channel to detect submarines as they pass through.¹ The patrol barrier will consist of a linear array of detection devices across the channel. Thus, the searching region is a straight line; and P1 divides this line into 15 cells, as shown in Figure 2.1. P1 would like to determine an optimal allocation of detection devices to maximize the probability of detecting P2.

Each detection device has a probability of detection versus range curve as given in Figure 2.2.² The payoff matrix can now be constructed from Figure 2.2. For example, if P1 searches cell 5 and P2 hides in cell 8, then the range is three cells and from Figure 2.2 $p_{52} = 0.367$. The complete P matrix is given in Table 2.1.

Now we have all of the information required to solve the n-cell game. The value and optimal strategies for this example were found by solving linear program (4) with the data of Table 2.1. A CDC 1604 computer and a standard linear programming code were used to effect the

¹This type of situation was encountered in the Bay of Biscay during World War II, Sternhell and Thorndike [1].

²This is a "typical" curve according to Morse and Kimball [1].

computations. P1 has the unique optimal strategy displayed in Table 2.2 and plotted in Figure 2.3. P2 has two optimal extreme point strategies, and they also appear in Table 2.2 and Figure 2.3.

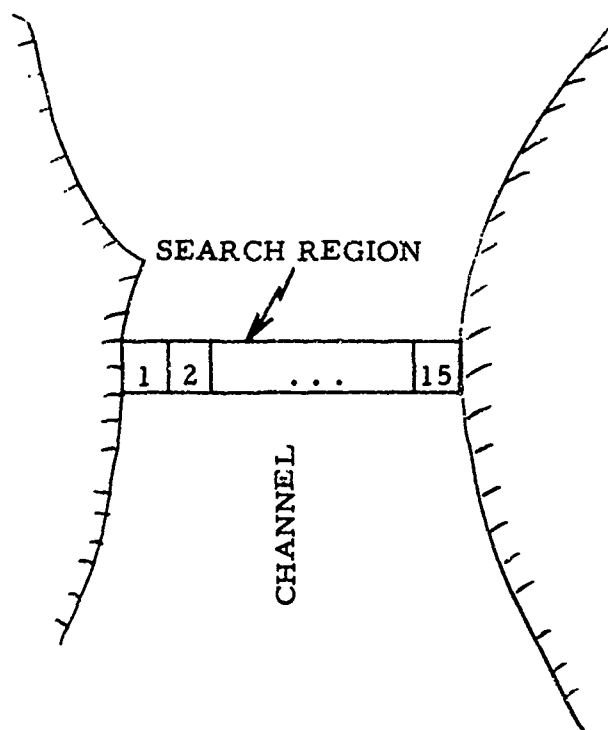
Let Y_1 and Y_2 denote P2's optimal extreme point strategies.

Then from linear programming theory, the strategy

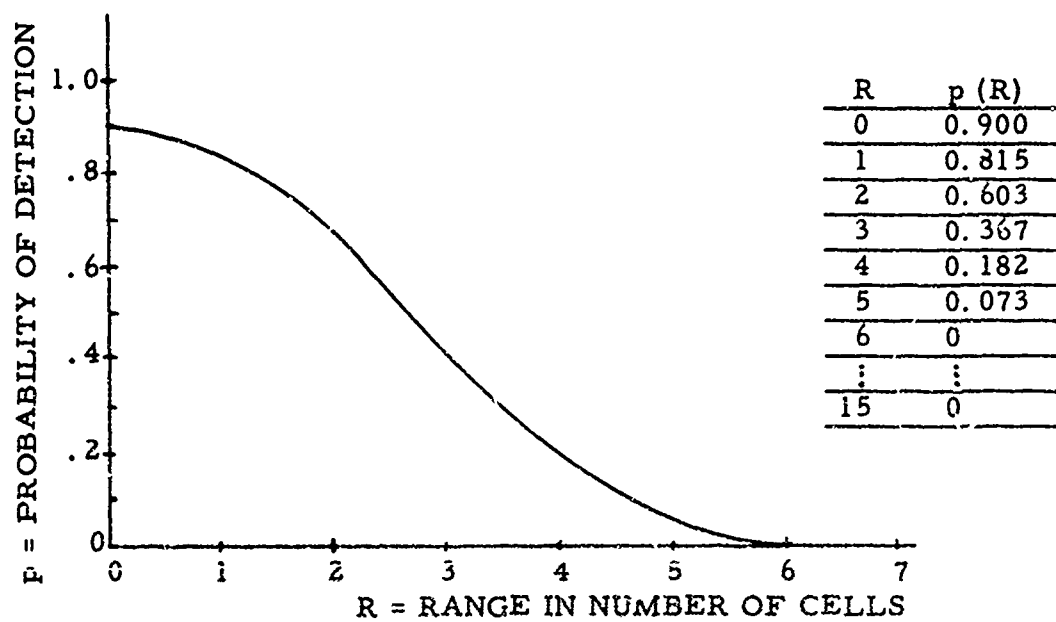
$$(6) \quad Y = \lambda Y_1 + (1 - \lambda) Y_2 \quad 0 \leq \lambda \leq 1$$

is also optimal. From the symmetry of the payoff matrix, we might expect P1 and P2 to have symmetric optimal strategies about the middle cell (cell 8). Indeed, P1's optimal strategy is symmetric about cell 8 and for $\lambda = \frac{1}{2}$ in equation (6); P2 also has a symmetric optimal strategy about cell 8.

Notice that a unit of P1's effort in cell 1 or 15 has only one-half the probability of detecting P2 as a unit in cell 8. However, about sixty-three percent of P1's effort is assigned to cells near the ends of the search region (cells 2 and 14). Then, in a sense, P1 compensates for the decreased effectiveness per unit in the end cells by assigning a large percentage of effort to these cells.



The Search Region and Cells
Figure 2.1



Probability of Detection vs. Range
Figure 2.2

PLAYER 2'S PURE STRATEGIES

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

1 .900 .815 .603 .367 .182 .073

2 .815 .900 .815 .603 .367 .182 .073

3 .603 .815 .900 .815 .603 .367 .182 .073

4 .367 .603 .815 .900 .815 .603 .367 .182 .073

5 .182 .367 .603 .815 .900 .815 .603 .367 .182 .073

6 .073 .182 .367 .603 .815 .900 .815 .603 .367 .182 .073

7 .073 .182 .367 .603 .815 .900 .815 .603 .367 .182 .073

8 .073 .182 .367 .603 .815 .900 .815 .603 .367 .182 .073

9 .073 .182 .367 .603 .815 .900 .815 .603 .367 .182 .073

10 .073 .182 .367 .603 .815 .900 .815 .603 .367 .182 .073

11 .073 .182 .367 .603 .815 .900 .815 .603 .367 .182

12 .073 .182 .367 .603 .815 .900 .815 .603 .367

13 .073 .182 .367 .603 .815 .900 .815 .603

14 .073 .182 .367 .603 .815 .900 .815 .900 .815

15 .073 .182 .367 .603 .815 .900 .815 .900

PLAYER 1'S PURE STRATEGIES

Player 1's Payoff Matrix
Table 2.1

j	y_j
1	0.258
5	0.069
6	0.165
10	0.234
15	0.274

all other $y_j = 0$

P2's Optimal Basic Strategies

j	y_j
1	0.274
6	0.234
10	0.165
11	0.069
15	0.258

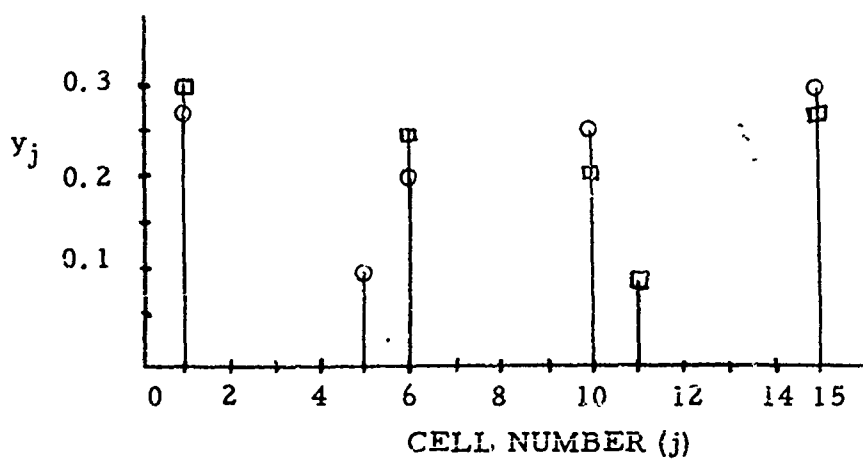
all other $y_j = 0$

j	x_j
2	0.317
6	0.099
8	0.168
10	0.099
14	0.317

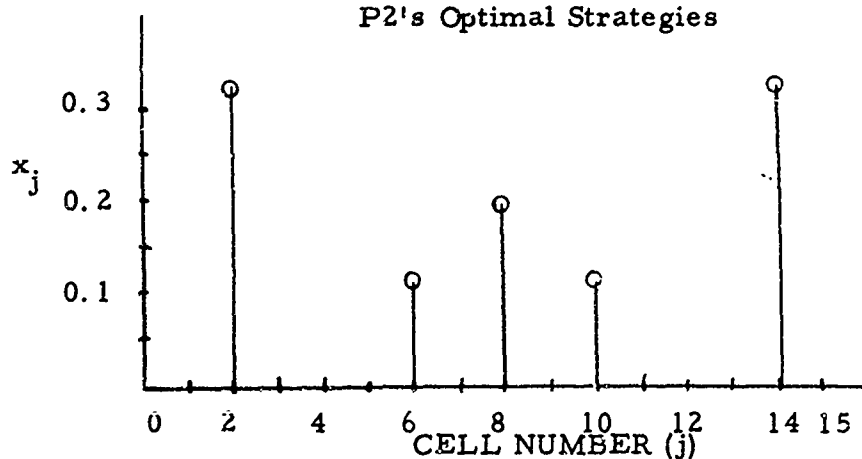
all other $x_j = 0$

P1's Optimal Strategy

Optimal Strategies
Table 2.2



P2's Optimal Strategies



P1's Optimal Strategy

Plots of the Optimal Strategies
Figure 2.3

Other tactical examples which fit the n -cell model are also immediately available. First, the search region may be a rectangular array of cells. In this case, an optimal strategy for P1 is likely to require P1 to play those cells on the edges of the search region with higher probability than the center cells. Intuitively this would prevent P2 from hiding in the edge cells where the detection capability is lower than in the center cells.

We might also study situations where the search region is three dimensional. The effect of the depth of P2 on the probability of detection can thereby be taken into account. If P1 is using surface detection devices then, of course, P1's pure strategies would include only the surface cells, and P2 could choose any cell in the three-dimensional region. This is a slight variation on the n -cell game where P1 and P2 do not have identical sets of pure strategies.

Next, we extend the n -cell game to include two or more types of detection devices for P1. For example, the searcher may have aircraft and ships available and the probability of detection verses range curve may differ significantly between ships and aircraft. For the sake of discussion, suppose that only two types of detection devices are to be used. Call these devices type 1 and type 2. Now the detection probability of interest is the joint conditional probability p_{ikj} . Where p_{ikj} is the probability that P1 detects P2 given type k is located in cell i ($i = 1, \dots, n$),

type 2 is located in cell k ($k = 1, \dots, n$) and P2 hides in cell j ($j = 1, \dots, n$). This game can be placed in the framework of the n -cell game by simply changing the pure strategies for P1. Let a pure strategy for P1 be the two-tuple (i, k) where i denotes the location of type 1 and k denotes the location of type 2. Now we may construct the payoff matrix and solve for the optimal strategies and value by reference to linear programs (4) and (5). This example may be extended to handle more than two types of equipment.

2.5 Extension to a Constrained Game

We extend the n-cell game to accommodate the following types of situations.

1. A player has information on the cells which his opponent can choose.
2. A player restricts his own choice of cells.

The above statements are necessarily broad to include a variety of tactical problems. Some of these problems are outlined below. As we shall see, cases 1 and 2 are formally embodied by the elegant notion of a constrained game due to Charnes [1].

We discuss 1 for information which P1 may have on P2's location. Analogous statements hold when P2 has information on P1. Now suppose P1 obtains a contact with P2 and subsequently loses the contact. Then P1 knows that P2 must be located in some subset I of the set of all cells, where I is determined by the position of the last contact and the elapsed time since the contact. Hence, the following constraints on P2's strategies are obtained:

$$y_j = 0 \text{ for } j \notin I$$

More detailed constraints on P2's strategies may also be written.

Morse and Kimball [1] give a theoretical probability distribution of P2's location as a function of elapsed time since the last contact. From this distribution, we can calculate the bounds L_j and U_j with

$$(7) \quad L_j \leq y_j \leq U_j \quad j = 1, \dots, n$$

The above restrictions may also arise from intelligence reports or P1's apriori estimates of P2's location.

We discuss two extreme cases of information for P1:

1. perfect information,
2. no information.

In case 1, P1 knows the strategy which P2 will employ. Of course, P1 will then have an optimal pure strategy. Furthermore, perfect information corresponds to $L_j = U_j$, $j = 1, \dots, n$, in equation (7). Case 2 is the unconstrained n -cell search game. Here we have $L_j = 0$, $U_j = 1$, $j = 1, \dots, n$, in equation (7). In many ASW situations, the information which is available will be between these two extremes. These cases yield to a constrained game formulation.

A situation where P1 may restrict his own choice of cells is when he has a secondary military objective in addition to the primary objective of detecting P2. For instance, P1 may wish to provide at least a certain level of protection for some set of cells I in the search region because there is a convoy in this set of cells. Then constraints of the following form arise:

$$\sum_{i \in I} x_i \geq c_I$$

where c_I is the desired level of minimum protection.

We have discussed a few situations in which constraints on the players' strategies arise naturally. These situations and others are included by the following sets of constraints on P1's and P2's strategies respectively

$$(8) \quad \sum_{i=1}^n x_i c_{is} \leq c_s \quad s = 1, \dots, S$$

$$(9) \quad \sum_{j=1}^n b_{rj} y_j \geq b_r \quad r = 1, \dots, R$$

The following constrained formulation due to Charnes [1] is employed to deal with the types of tactical situations under consideration. It includes the formulated constraints (8) and (9).

$$\begin{aligned} \text{Max } v + \sum_r z_r b_r & \qquad \text{Min } u + \sum_s c_s w_s \\ v + \sum_r z_r b_{rj} - \sum_i x_i p_{ij} & \leq 0 \qquad u + \sum_s c_{is} w_s - \sum_j p_{ij} y_j \geq 0 \\ (10) \quad \sum_i x_i & = 1 \qquad (11) \quad \sum_j y_j = 1 \\ \sum_i x_i c_{is} & \leq c_s \qquad \sum_j b_{rj} y_j \geq b_r \\ z_r, x_i & \geq 0 \qquad w_s, y_j \geq 0 \end{aligned}$$

Let quantities with a "hat" over them denote part of an optimal solution to (10) and (11). Charnes obtained the following results:

$$(12) \quad \hat{v} + \sum_r \hat{z}_r b_r = \sum_i \sum_j \hat{x}_i p_{ij} \hat{y}_j = \hat{u} + \sum_s c_s \hat{w}_s$$

$$(13) \sum_i \sum_j x_i p_{ij} \hat{y}_j \leq \sum_i \sum_j \hat{x}_i p_{ij} \hat{y}_j \leq \sum_i \sum_j \hat{x}_i p_{ij} y_j \quad \text{all strategies } X \text{ and } Y$$

Equations (12) and (13) establish the existence of a value and optimal strategies for the constrained game. Of course, the value and optimal strategies may be computed from the above linear programs.

We compare the constrained n-cell game to the unconstrained game. Suppose that P1 can impose constraints on P2's strategies but there are no constraints on P1's strategies. Then the value of the constrained game is no smaller than the unconstrained value. This fact follows from problems (10) and (11), since the unconstrained value can be attained by the objective function of (10) with all $z_r = 0$. Hence, P1 can always increase the probability of detecting P2 if he can determine constraints on P2's strategies without imposing constraints on his own strategies. Analogous remarks hold for player 2. See Charnes and Cooper [5] and Sakaguchi [1], [2], for more details on this subject.

Finally, we discuss an adaptive manner of employing constrained games. If P1 obtains additional information on P2's location in a particular play of the game, then P1's optimal strategy is likely to change for the next play. On the other hand, if no additional information is obtained, then P1 will have the same optimal strategy for the next play. Of course, these remarks also hold for P2's optimal strategies. In the constrained version of the n-cell game, the players can choose their optimal strategies adaptively with the optimal strategies for a particular

play depending on the actual information which is obtained in preceding plays. However, the constrained model does not include the evaluation of the possible future consequences of a strategy, and thus the game is non-sequential in nature.

2.6 A Special Case: Negligible Radius of Detection

In section 2.2 we mentioned that all of the existing search models assume a negligible radius of detection. This assumption for the n -cell game requires P to be a diagonal matrix. For then, P_1 can detect P_2 with non-zero probability only if P_1 searches the cell in which P_2 is hiding. This special case was first proposed and solved by von Neumann [1]. We give an alternate derivation which is equivalent to von Neumann's proof.

We assume that P is a diagonal matrix, i. e., $p_{ij} = 0$ for $i \neq j$. We also assume without loss of generality that $p_{ii} > 0$, $i = 1, \dots, n$. For if $p_{ii} = 0$ for some i , then the game has a saddle point in pure strategies, and we exclude this trivial case. Under the above two assumptions, the dual linear programs (4) and (5) which characterize the solution become:

Max v

$$v - x_i p_{ii} \leq 0$$

$$(14) \quad \sum x_i = 1$$

$$x_i \geq 0$$

Min u

$$u - p_{jj} y_j \geq 0$$

$$(15) \quad \sum_j y_j = 1$$

$$y_j \geq 0$$

Now we find an analytic solution to (14) and (15). Let $\hat{v}, \hat{X} = (\hat{x}_1, \dots, \hat{x}_n)$ be an optimal solution to (14). From the inequality constraints in (14), \hat{v} must satisfy

$$(16) \quad \hat{v} = \min_i \max_x x_i p_{ii} = \min_i \hat{x}_i p_{ii}$$

Notice that for \hat{x}_i to be optimal we must have

$$(17) \quad p_{ii} \hat{x}_i = c \quad i = 1, \dots, n$$

For if (17) does not hold, then we can construct a strategy which yields a larger \hat{v} . From (17)

$$\hat{x}_i = \frac{c}{p_{ii}} \quad i = 1, \dots, n$$

where c is chosen to insure $\sum_{i=1}^n \hat{x}_i = 1$, i. e.,

$$\frac{1}{c} = \sum_{i=1}^n \frac{1}{p_{ii}}$$

From (16) and (17) we have

$$\hat{v} = c$$

A similar analysis shows that \hat{X} is also an optimal strategy for player 2.

Returning to the numerical example of section 2.4, we find that if P is a diagonal matrix then \hat{X} and \hat{Y} are uniform distributions. This result clearly points out the limited applicability to ASW of negligible radius of detection models.

2.7 Formulation of the Row-Column Search Game

For the sake of discussion, we call the next search game of interest the row-column search game. This game is similar to the n -cell game in that (1) the searching region is divided into cells, (2) the payoff is a probability of detection, and (3) it is a two-person zero-sum game with P1 the maximizing player. The row-column game differs from the n -cell game in the manner in which player 1 conducts the search, and therefore different tactical situations are represented.

Now the game is formulated. As before, a pure strategy for P2 is a cell in which to hide, but now the cells are doubly indexed (i, j) $i = 1, \dots, m, j = 1, \dots, n$. A pure strategy for P1 is the choice of an index i or j . If the cells are thought of as positions in an $m \times n$ matrix, then P2 chooses a position and P1 chooses a row or column. These choices are made simultaneously and constitute a play of the game.

The row-column game is especially useful for studying certain types of search situations. A typical situation occurs when searching is conducted in sweeps and the speed of the searching craft is substantially faster than the speed of the submarine. For then P1 can search an entire row or column while P2 stays in one cell. One example of the row-column game is then search in "sweeps" by aircraft. On the other hand, the n -cell model is better suited to search by slow craft, such as ships, because each player only chooses one cell in a play of the game.

As in the n -cell game, the payoff function will be the probability that P1 detects P2 (the probability of detection). However, for the present game, the probability of detection is taken to be the product of the probability that P1 contacts P2 (contact probability) and the conditional probability that P1 identifies the contact as P2 given a contact has been made (identification probability). We could simply postulate a detection probability with the understanding that both contact and identification probabilities are included. However, we postulate the two probabilities separately to exhibit certain predominant features of this game.

If P2 hides in cell (i, j) , then P1 may contact P2 with positive probability only if P1 searches row i or column j . But, the contact probability itself may vary along a particular row or column. This variation may be due to differences in water temperature, salinity, bottom conditions, and a host of other factors. Now the identification probability may depend on the row or column searched due to the detection equipment or crew proficiencies of the search craft which are available for a particular row or column.

To formally write down the payoffs, suppose that P2 hides in cell (i, j) and P1 searches row i , then the contact probability is p_{ij} and the identification probability is a_i and, therefore, the probability that P1 detects P2 is $a_i p_{ij}$. Similarly, if P2 hides in cell (i, j) and P1

searches column j , the contact probability is q_{ij} and the identification probability is b_j . The conditional probability of detection is now $b_j q_{ij}$. Of course, the above probabilities are defined for all i and j .

From the above definitions of the payoff elements, we construct player 1's payoff matrix, Table 2.3.

PLAYER 2's PURE STRATEGIES

	(1, 1) (1, 2) ... (1, n)	(2, 1) (2, 2) ... (2, n)	(m, 1) (m, 2) ... (m, n)
1	$a_{11} p_{11} a_{12} p_{12} \dots a_{1n} p_{1n}$		
2		$a_{21} p_{21} a_{22} p_{22} \dots a_{2n} p_{2n}$	
\vdots			
m			$a_{m1} p_{m1} a_{m2} p_{m2} \dots a_{mn} p_{mn}$
1	$b_{11} q_{11}$	$b_{12} q_{12}$	$b_{1n} q_{1n}$
2		$b_{21} q_{21}$	$b_{22} q_{22}$
\vdots			
n			$b_{n1} q_{n1}$

PLAYER 2's PURE STRATEGIES

All blank elements in the above matrix are zeros.

Player 1's Payoff Matrix
Table 2.3

2.8 Solution and Reduction to a Dyadic Model

Now that we have the payoff matrix, the game may be readily solved. Let y_{ij} be the probability that P2 chooses cell (i, j) $i = 1, \dots, m$, $j = 1, \dots, n$. Also, let u_i be the probability that P1 chooses row i ($i = 1, \dots, m$); and let v_j be the probability that P1 chooses column j ($j = 1, \dots, n$). Let Y be the $mn \times 1$ vector $Y = (y_{11}, \dots, y_{ij}, \dots, y_{mn})$; and let $U = (u_1, \dots, u_m)$ and $V = (v_1, \dots, v_n)$. The following dual linear programs¹ which characterize the game solution are obtained directly from the payoff matrix

$$\begin{array}{ll}
 \text{Max } \rho & \text{Min } \xi \\
 \rho - a_i p_{ij} u_i - b_j q_{ij} v_j \leq 0 & \xi - \sum_j a_i p_{ij} y_{ij} \geq 0 \\
 (18) \quad \sum_i u_i + \sum_j v_j = 1 & (19) \quad \xi - \sum_i b_j q_{ij} y_{ij} \geq 0 \\
 u_i, v_j \geq 0 & \sum_{i=1}^m \sum_{j=1}^n y_{ij} = 1 \\
 & y_{ij} \geq 0
 \end{array}$$

Let $\hat{\rho}$, $\hat{\xi}$, (\hat{U}, \hat{V}) , \hat{Y} be part of an optimal solution to (18) and (19).

Then from Charnes [1], (\hat{U}, \hat{V}) and \hat{Y} are optimal strategies for players 1 and 2 respectively and $\hat{\rho}$ is the value.

To obtain additional insights into the row-column game, we

¹Due to Charnes [1].

transform problem (19) to a dyadic model. We make the following two assumptions and show that they do not result in loss of generality.

$$(i) \quad p_{ij}, q_{ij} > 0 \quad \text{all } i, j$$

$$(ii) \quad a_i, b_j > 0 \quad \text{all } i, j$$

Now a_i and b_j are probabilities; therefore, $0 \leq a_i \leq 1$, $0 \leq b_j \leq 1$. Since the a_i and b_j are non-negative, the addition of a large positive constant to every element of the payoff matrix yields a game with all positive p_{ij} and q_{ij} and the optimal strategies are not altered, von Neumann and Morgenstern [1]. Therefore, assumption (i) does not result in loss of generality. Suppose (i) is satisfied and consider assumption (ii). We exclude the trivial case; all $a_i = b_j = 0$, and assume some a_i or b_j are positive. Every row in the payoff matrix with $a_i = 0$ or $b_j = 0$ is dominated by a row with positive a_i or b_j . Thus, the rows with zero a_i or b_j may be deleted from the payoff matrix and assumption (ii) does not result in loss of generality. From assumptions (i) and (ii) it follows immediately that the optimal objective functions for (18) and (19) must satisfy $\hat{\rho}, \hat{\xi} > 0$.

The desired transformation for problem (18) is

$$(20) \quad u'_i = \frac{a_i u_i}{\rho} \quad i = 1, \dots, m; \quad v'_j = \frac{b_j v_j}{\rho} \quad j = 1, \dots, n$$

and for problem (19)

$$(21) \quad y'_{ij} = \frac{y_{ij}}{\xi} \quad i = 1, \dots, m; \quad j = 1, \dots, n$$

These transformations yield the following dual pair of linear programs:

$$\begin{aligned}
 (22) \quad & \text{Min } \sum_i \frac{1}{a_i} u'_i + \sum_j \frac{1}{b_j} v'_j \\
 & u'_i p_{ij} + v'_j q_{ij} \geq 1 \\
 & u'_i, v'_j \geq 0 \\
 (23) \quad & \text{Max } \sum_{i,j} y'_{ij} \\
 & \sum_j p_{ij} y'_{ij} \leq \frac{1}{a_i} \\
 & \sum_i q_{ij} y'_{ij} \leq \frac{1}{b_j} \\
 & y'_{ij} \geq 0
 \end{aligned}$$

Since $\hat{p}, \hat{\xi} > 0$, equations (20) and (21) establish a one-to-one correspondence between optimal solutions to (18) and (22) and (19) and (23). Let "hats" on the variables denote an optimal solution to (22) and (23).

Then from (20) and (21) we have

$$(24) \quad \hat{u}_i = \frac{\hat{p} \hat{u}'_i}{a_i}, \quad \hat{v}_j = \frac{\hat{p} \hat{v}'_j}{b_j}, \quad \hat{p} = \frac{1}{\sum_i \frac{1}{a_i} \hat{u}'_i + \sum_j \frac{1}{b_j} \hat{v}'_j}$$

$$(25) \quad \hat{y}_{ij} = \hat{\xi} \hat{y}'_{ij}, \quad \hat{\xi} = \frac{1}{\sum_{i,j} \hat{y}'_{ij}}$$

Problem (23) is a dyadic model as defined by Charnes and Cooper [5]. Actually, (23) is not the most general dyadic model but it is substantially more general than the distribution (transportation) model. Special computational techniques are available to solve dyadic problems but we do not dwell on them here. The reader is referred to Charnes and Cooper [5].

2.9 Special Cases

Certain tactical situations may be formulated as special cases of (22) and (23). One of these cases is when

$$(26) \quad p_{ij} = q_{ij} \quad \text{all } i, j$$

Equation (26) implies that the contact probability for cell (i, j) is independent of whether P1 searches row i or column j, i. e., the direction in which the sweep is made is immaterial. Suppose (26) is satisfied, and let

$$(27) \quad w_{ij} = p_{ij} v'_{ij} \equiv q_{ij} v'_{ij}$$

then problem (23) is transformed by (27) to yield the following dual pair of linear programs:

$$\begin{array}{ll}
 \text{Min } \sum \frac{1}{a_i} u'_i + \sum \frac{1}{b_j} v'_j & \text{Max } \sum_{i,j} \frac{1}{p_{ij}} w_{ij} \\
 u'_i + v'_j \geq \frac{1}{p_{ij}} & (29a) \quad \sum_j w_{ij} \leq \frac{1}{a_i} \\
 u'_i, v'_j \geq 0 & (29b) \quad \sum_i w_{ij} \leq \frac{1}{b_j} \\
 & w_{ij} \geq 0
 \end{array}
 \quad (28)$$

An additional simplification of the distribution problem (29) can be obtained. Summing out over the constraints (29a) and (29b) respectively, we get

$$\sum_{i,j} w_{ij} \leq \sum_i \frac{1}{a_i}, \quad \sum_{i,j} w_{ij} \leq \sum_j \frac{1}{b_j}$$

Since $\frac{1}{p_{ij}} > 0$, the same optimal solution obtains if we replace (29a) or (29b) by equalities as follows:

(i) If $\sum_i \frac{1}{a_i} < \sum_j \frac{1}{b_j}$ replace (29a) by equalities.

(ii) If $\sum_i \frac{1}{a_i} > \sum_j \frac{1}{b_j}$ replace (29b) by equalities.

(iii) If $\sum_i \frac{1}{a_i} = \sum_j \frac{1}{b_j}$ replace both (29a) and (29b) by equalities.

Under the indicated assumption (26), the row-column search game has been reduced to a distribution model. We give a tactical example of this model in the next section.

A final simplification of the most general game obtains if, in addition to (26), we assume that $a_i = b_j = 1$ and $m = n$. Of course, these assumptions mean that the identification probabilities are one and the search region is divided into an equal number of rows and columns. Now (28) and (29) are reduced to the following dual linear programs.

$$\begin{aligned}
 \text{Min } & \sum_i u'_i + \sum_j v'_j \\
 (30) \quad & u'_i + v'_j \geq \frac{1}{p_{ij}} \\
 & u'_i, v'_j \geq 0
 \end{aligned}$$

$$\begin{aligned}
 \text{Max } & \sum_{i,j} \frac{1}{p_{ij}} w_{ij} \\
 (31) \quad & \sum_j w_{ij} \leq 1 \\
 & \sum_i w_{ij} \leq 1 \\
 & w_{ij} \geq 0
 \end{aligned}$$

From statement (iii) above, the constraints in (31) are satisfied as equalities by an optimal solution. But, the equality form of (31) is the well-known assignment problem. This equivalent assignment problem was first obtained by von Neumann [1]. Our proof of equivalence is considerably more direct than his, due to the linear programming characterization of a matrix game which is now available. This completes the transformations of the row-column game to dyadic-type models.

2.10 Tactical Example

We conclude this chapter with an example of the row-column search game. In this example, the search will be conducted by a single aircraft. The search region of interest is partitioned into four rows and six columns, and the aircraft searches in row or column sweeps. We assume that when P2 hides in cell (i, j) the contact probabilities depend on (i, j) but not on whether P1 searches row i or column j . Thus, we have $p_{ij} = q_{ij}$ all i, j , and this game is equivalent to the distribution problem (29). The contact probabilities and the identification probabilities are given in Table 2.4.

To apply standard methods to solve (29), we convert it to the standard equality form. For this example $\sum_i \frac{1}{a_i} > \sum_j \frac{1}{b_j}$, thus the inequalities (29b) are automatically satisfied as equalities by the optimal solution to (29). Therefore, we simply adjoin a dummy column to (29) to obtain the equivalent standard distribution problem. This standard form and the optimal solutions are given in Table 2.5. Finally, the optimal solutions are converted to strategies and tabulated in Table 2.6.

	j					
	1	2	3	4	5	6
1	0.5	0.4	0.5	0.6	0.6	0.7
2	0.4	0.3	0.6	0.7	0.8	0.5
3	0.5	0.7	0.7	0.8	0.9	0.6
4	0.6	0.6	0.5	0.7	0.8	0.6
	1/3	1/3	1/2	1	1	1/2
						a_i b_j

The contact probabilities (p_{ij}) are given in the first four rows and the first six columns. The identification probabilities are given in the last row and the last column.

Contact and Identification Probabilities
Table 2.4

$u_i \backslash v_j$	2	2.83	?	1.67	1.67	1.67	0	$1/a_i$
0	2	2.5	2 (6)	1.67 (1)	1.67 (1)	1.43	0	2
.5	2.5 (2)	3.33 (3)	1.67	1.43	1.25	2	0	5
0	2 (1)	1.43	1.43	1.25	1.11	1.67 (2)	0 (1)	4
0	1.67	1.67	2 (2)	1.43	1.25	1.67	0 (1)	3
$1/b_j$	3	3	2	1	1	2	2	

The Distribution Tableau and an Optimal Solution

$u_i \backslash v_j$	2	2.83	2	1.67	1.67	1.67	0	$1/a_i$
0			(6)	(1)	(1)			2
.5	(2)	(3)						5
0	(1)					(1)	(2)	4
0			(2)			(1)		3
$1/b_j$	3	3	2	1	1	2	2	

An Alternate Optimal Solution

Optimal Tableaus
Table 2.5

The transformation formulas to obtain optimal strategies from the optimal solutions of Table 2.4 are:

$$\frac{1}{\hat{s}} = \sum_{i,j} \frac{1}{p_{ij}} \hat{w}_{ij} = 27.68$$

$$\hat{y}_{ij} = \hat{s} \left(\frac{\hat{w}_{ij}}{p_{ij}} \right)$$

$$\hat{v}_j = \frac{\hat{v}'_j \hat{s}}{b_j} \quad \hat{u}_i = \frac{\hat{u}'_i \hat{s}}{a_i}$$

Optimal Strategies for P2

(i, j)	\hat{y}_{ij}	\hat{y}'_{ij}
(1, 4)	0.060	0.060
(1, 5)	0.060	0.060
(2, 1)	0.181	0.181
(2, 2)	0.361	0.361
(3, 1)	0.073	0.073
(3, 6)	0.120	0.060
(4, 3)	0.145	0.145
(4, 6)	-----	0.060

all other $\hat{y}_{ij} = 0$, all other $\hat{y}'_{ij} = 0$

Optimal Strategy for P1

$$\hat{u}_2 = 0.092$$

$$\hat{v}_1 = 0.217$$

$$\hat{v}_2 = 0.306$$

$$\hat{v}_3 = 0.146$$

$$\hat{v}_4 = 0.060$$

$$\hat{v}_5 = 0.060$$

$$\hat{v}_6 = 0.121$$

all other $\hat{u}_i = \hat{v}_j = 0$

Optimal Strategies
Table 2.6

CHAPTER III - SEQUENTIAL N-CELL GAME

3.1 Results

In this chapter, we formulate and solve a sequential search game.

This game consists of a sequence of moves, and on each move the players are faced with an n -cell search game. We consider, in turn, both a finite and infinite number of moves. In essence, on each move the players simultaneously choose a strategy in an n -cell search game and thereby determine an immediate payoff and, in addition, a probability that the n -cell game is played again. Hence, a sequence of n -cell games is played. We show how to find optimal strategies for both the finite and infinite games which minimax the expected accumulated payments.

For the finite game, we show how to characterize the value and optimal strategies in a recursive manner. In this way, we can compute the solution by linear programming methods.

A characterization of the solution of the infinite game results in a non-linear programming problem. However, if one variable is treated as a parameter, the resulting problem is a linear program. We show how to change this parameter in successive solutions to the linear program and thereby approximate a solution to the non-linear problem.

We discuss in detail two particular payoffs which are meaningful for ASW purposes. One of these payoffs reduces our characterization of the infinite game to a linear programming problem. Finally, examples are given, and we compare the sequential n -cell game to the non-sequential n -cell game.

3.2 Formulation of the Finite Game

First, we discuss the elements of the finite game, and then we proceed with the mathematical formulation. A play of the game consists of, at most, a finite number (N) of moves. On each move, when the game has not terminated, the players are faced with a two-person zero-sum game. In our formulation, we shall use the n -cell game as the two-person zero-sum game for each move. When the players move, they each choose a strategy which determines a zero-sum payoff from player 2 to player 1 and a probability that the game terminates before the next move. We wish to find an optimal strategy for each player which minimizes the expected accumulated payments received by player 1.

For ASW purposes, we consider two particular payoffs. As in the n -cell game, the payoff for each move may be the probability that P_1 detects P_2 during the move. Then, as we shall see, the expected accumulated payment received by P_1 is the probability that P_1 detects P_2 . The other payoff considered is the time taken by one move. Here, P_1 receives a payoff of one time unit regardless of the strategies chosen, and the expected accumulated payment is the expected duration of the game. In the following formulation, we use the generic term payoff to accommodate both of the above tactical payoffs and others as well.

The recursive optimization technique which we will propose has also been discussed by other authors. Kuhn [1] (1953) gave his

theorem on games of perfect recall which paved the way for further work. Shapley [1] (1953) was the first to use a recursive optimization technique for this type of game, although he did not deal with the finite case. Later contributions were made by Bellman [1] (1957), Zachrisson [1] (1964), and Denardo [1] (1965). However, the development given here differs in content and detail from the above cited works.

We assume that a search region is given and that the region is divided into n cells numbered $i = 1, \dots, n$. We also assume that a pure strategy for each player on each move corresponds to the choice of a cell. From a tactical standpoint, a pure strategy for P1 (the searcher) is a cell to search and a pure strategy for P2 (the hider) is a cell in which to hide. Notice that we have assumed that the same set of pure strategies is available for each player on each move. We have taken this assumption for notational convenience; it could be relaxed. We further assume that a play of the game consists of, at most, N moves; and we number the possible moves $r = 1, 2, \dots, N$. On each move, the players choose their strategies simultaneously and the moves are made sequentially. Unless otherwise stated, we assume that P1 is the maximizing player.

The payoffs and continuation probabilities are now specified. Suppose that P1 searches cell i and P2 hides in cell j on move r . Then the payoff from P2 to P1 is

$$a_{ij}(r) \quad i, j = 1, \dots, n$$

$$r = 1, \dots, N$$

Also, when P1 searches cell i and P2 hides in cell j on move r , the game continues until move $r + 1$ with probability.

$$p_{ij}(r) \quad i, j = 1, \dots, n$$

$$r = 1, \dots, N - 1$$

We let A_r be the $n \times n$ matrix $A_r = (a_{ij}(r))$ and P_r the $n \times n$ matrix $P_r = (p_{ij}(r))$. Hence, A_r is P1's payoff matrix for move r and P_r is the matrix of continuation probabilities for move r .

Next, we consider strategies for the players. We have assumed that the continuation probability and payoff depend only on the choices available for a particular move. It follows that the game is one of perfect recall as defined by Kuhn [1]. Kuhn's theorem for a game of perfect recall asserts that an optimal strategy for a particular move does not depend on preceding strategies. Hence, an optimal strategy for each particular move in our game is a mixed strategy over the alternatives available at that move. Kuhn calls this type of strategy a "behavior strategy". We restrict our attention to these mixed strategies without loss of generality.

Let X_r and Y_r be mixed strategies over the alternatives available on move r for P1 and P2 respectively. Let $X = (X_1, \dots, X_N)$ be an N -tuple of mixed strategies for P1, with X_r being the mixed strategy

for move r . Similarly, let $Y = (Y_1, \dots, Y_N)$ be P2's game strategy.

Now we define the following sets of strategies

$$\mathbf{X}_r = \{X_r\}, \mathbf{Y}_r = \{Y_r\}, \mathbf{X} = \{X\}, \mathbf{Y} = \{Y\}$$

From the above discussion, \mathbf{X} and \mathbf{Y} contain optimal game strategies for P1 and P2 respectively. These optimal game strategies are optimal with respect to the set of all possible strategies.

We will write the total expected payoff for P1 in terms of fixed strategies $X \in \mathbf{X}$, $Y \in \mathbf{Y}$ and the given information. If P1 chooses the strategy X_r for move r and P2 chooses Y_r , then the payoff to P1 for move r is

$$X_r^t A_r Y_r \quad r = 1, \dots, N$$

and the game continues until move $r + 1$ with probability

$$X_r^t P_r Y_r \quad r = 1, \dots, N-1$$

Now the expected payoff to P1 for move r is the product of the probability that the game continues until move r and the payoff for move r

$$X_r^t A_r Y_r \prod_{h=1}^{r-1} X_h^t P_h Y_h \quad r = 2, 3, \dots, N$$

The expected accumulated payoff for N moves, $v_1(X, Y)$, is the sum of the above expected payoffs for each move.

$$(1) \quad v_1(X, Y) = X_1^t A_1 Y_1 + \sum_{r=2}^N X_r^t A_r Y_r \prod_{h=1}^{r-1} X_h^t P_h Y_h$$

Since the game has a finite number of moves and a finite number of strategies, it must have a value and optimal strategies, von Neumann and Morgenstern [1]. Recall that the sets \mathcal{X} and \mathcal{Y} contain optimal game strategies. Therefore, the function $v_1(X, Y)$ has at least one saddle point over the sets \mathcal{X} and \mathcal{Y} . We propose a recursive optimization technique to find the saddle points of $v_1(X, Y)$.

3.3 Recursive Solution

In this section, we show how to compute the minimax of equation (1) by a recursive technique. Let \bar{X}_r and \bar{Y}_r denote the sequences of strategies

$$\begin{aligned}\bar{X}_r &= (X_r, X_{r+1}, \dots, X_N) \\ \bar{Y}_r &= (Y_r, Y_{r+1}, \dots, Y_N)\end{aligned}\quad r = 1, \dots, N$$

Of course, we have $\bar{X}_1 = X$, and $\bar{Y}_1 = Y$. We rewrite equation (1) and define the functions $v_r(\bar{X}_r, \bar{Y}_r)$ by

$$(2) \quad v_r(\bar{X}_r, \bar{Y}_r) = X_r^t A_r Y_r + (X_r^t P_r Y_r) v_{r+1}(\bar{X}_{r+1}, \bar{Y}_{r+1}) \quad r = 1, \dots, N$$

$$v_{N+1} \equiv 0$$

Now, $v_r(\bar{X}_r, \bar{Y}_r)$ may be interpreted as the expected accumulated payments received by P1 on the last $N - r + 1$ moves of the game.

It is intuitively clear from equation (2) that the value and optimal strategies may be computed recursively. We shall establish this fact.

We define $\hat{v}_1, \hat{X}_r, \hat{Y}_r$ by the following equations

$$(3) \quad \hat{v}_r = \max_{X_r \in X_r} \min_{Y_r \in Y_r} \left[X_r^t A_r Y_r + (X_r^t P_r Y_r) \hat{v}_{r+1} \right] \quad r = 1, \dots, N$$

$$\hat{v}_{N+1} \equiv 0$$

$$= \hat{X}_r^t A_r \hat{Y}_r + (\hat{X}_r^t P_r \hat{Y}_r) \hat{v}_{r+1}$$

The minimax theorem of von Neumann [2] establishes the existence of

$\hat{X}_r, \hat{Y}_r, \hat{v}_r$ for equation (3). The following theorem relates the solutions

of equation (3) to the solutions of the sequential game.

Theorem 1: \hat{v}_1 is the value of the sequential game, and $\hat{X} = (\hat{X}_1, \dots, \hat{X}_N)$, $\hat{Y} = (\hat{Y}_1, \dots, \hat{Y}_N)$ are optimal strategies for P1 and P2 respectively.

Proof - Since $v_1(X, Y)$ is the expected payoff function for the sequential game, a necessary and sufficient condition for \hat{v}_1 to be the value of the game and \hat{X} , \hat{Y} optimal strategies is

$$v_1(X, \hat{Y}) \leq \hat{v}_1 \leq v_1(\hat{X}, Y) \quad \text{all } X \in \mathbf{X} \text{ and } Y \in \mathbf{Y}$$

We shall show that this condition is satisfied by \hat{v}_1 , \hat{X} , \hat{Y} as defined by

(3). From (3) we have

$$(4) \quad \hat{X}_r^t A \hat{Y}_r + (\hat{X}_r^t P_r \hat{Y}_r) \hat{v}_{r+1} \leq \hat{v}_r \leq \hat{X}_r^t A Y_r + (\hat{X}_r^t P_r Y_r) v_{r+1},$$

$$\text{all } X_r \in \mathbf{X}_r, Y_r \in \mathbf{Y}_r$$

We begin an inductive argument

$$v_N(\hat{X}_N, Y_N) = \hat{X}_N^t A_N Y_N \geq \hat{v} \quad \text{all } Y_N$$

assume

$$v_{r+1}(\hat{\bar{X}}_{r+1}, \bar{Y}_{r+1}) \geq \hat{v}_{r+1} \quad \text{for some } r$$

By definition

$$v_r(\hat{\bar{X}}_r, \bar{Y}_r) = \hat{\bar{X}}_r^t A_r Y_r + (\hat{\bar{X}}_r^t P_r Y_r) v_{r+1}(\hat{\bar{X}}_{r+1}, \bar{Y}_{r+1}) \quad \text{all } \bar{Y}_r$$

By the inductive assumption and $\hat{\bar{X}}_r^t P_r Y_r \geq 0$

$$v_r(\bar{X}_r, \bar{Y}_r) \geq \hat{\bar{X}}_r^t A_r Y_r + (\hat{\bar{X}}_r^t P_r Y_r) \hat{v}_{r+1} \quad \text{all } \bar{Y}_r$$

From the above equation and equation (4)

$$v_r(\hat{X}_r, \bar{Y}_r) \geq \hat{v}_r \quad \text{all } \bar{Y}_r$$

Hence, by induction on r

$$\hat{v}_1 \leq v_1(\hat{X}, Y) \quad \text{all } Y \in Y$$

Similarly, we may establish

$$\hat{v}_1 \geq v_1(X, \hat{Y}) \quad \text{all } X \in X$$

therefore

$$v_1(X, \hat{Y}) \leq \hat{v}_1 \leq v_1(\hat{X}, Y) \quad \text{all } X \in X, Y \in Y$$

and the theorem is true.

We have established that the value and optimal strategies may be computed by means of equation (3).

For each fixed r in (3) we must solve an ordinary matrix game.

The game has the payoff matrix $A_r + \hat{v}_{r+1} P_r$ with \hat{v}_{r+1} known. As in Chapter 2, we draw on the following linear programming formulation of this game due to Charnes [1].

L.P. $(\hat{v}_{r+1}) \quad r = 1, \dots, N$

Max v_r

$$v_r - \sum_{i=1}^n x_{ir} [a_{ij}(r) + \hat{v}_{r+1} p_{ij}(r)] \leq 0 \quad j = 1, \dots, n$$

$$\sum_{i=1}^n x_{ir} = 1$$

$$x_{ir} \geq 0 \quad i = 1, \dots, n$$

From Charnes [1], an optimal solution to L. P. (\hat{v}_{r+1}) solves equation (3), i. e., it yields an optimal strategy \hat{X}_r for P1 on move r . Of course, an optimal solution to the dual of L. P. (\hat{v}_{r+1}) also yields an optimal strategy \hat{Y}_r for P2, and this strategy is available when the primal is solved.

The value and optimal strategies may be computed by the following method.

1. Set $\hat{v}_{N+1} = 0$.
2. Given \hat{v}_{r+1} , solve L. P. (\hat{v}_{r+1}) for an optimal solution \hat{v}_r , \hat{X}_r , and also obtain a dual optimal strategy \hat{Y}_r .
3. Return to step 2 until \hat{v}_1 is computed.

As we have shown, \hat{v}_1 is the value of the game and $\hat{X} = (\hat{X}_1, \dots, \hat{X}_N)$, $\hat{Y} = (\hat{Y}_1, \dots, \hat{Y}_N)$ are optimal game strategies for P1 and P2 respectively. In the next section, we give a simple example of the above method.

3.4 The Negligible Radius of Detection Assumption and an Example

We briefly examine the negligible radius of detection assumption for the sequential n-cell game. In section 2.6 we developed this assumption for the n-cell game and found the optimal strategies and value. Of course, some of the results obtained in section 2.6 will carry over directly to the present discussion. To take the negligible radius of detection assumption, we assume that A_r and P_r are diagonal matrices ($r = 1, \dots, N$). We denote the diagonal elements of A_r and P_r by a_{ir} and p_{ir} ($i = 1, \dots, n$) respectively. Under this assumption, if P1 and P2 both choose cell i on move r then the payoff to P1 is a_{ir} and the game continues until move $r + 1$ with probability p_{ir} ; otherwise, the payoff and continuation probability are zero. In terms of tactics, this model could be used in situations where P1 already has a contact with P2 and P1 may have a positive probability of maintaining the contact with P2 (the game may continue) only if P1 looks in the cell where P2 is hiding. Otherwise, P2 evades P1.

In this present special case, equation (3) is rendered

$$\hat{v}_r = \max_{X_r} \min_{Y_r} \left[\sum_{i=1}^n x_{ir} a_{ir} y_{ir} + \left(\sum_{i=1}^n x_{ir} p_{ir} y_{ir} \right) \hat{v}_{r+1} \right] \quad r = 1, \dots, N$$

$$\hat{v}_{N+1} = 0$$

From section 2.6,¹ the solution to the above equation with $r = N$ is

¹As in section 2.6, we assume without loss of generality that $a_{ir} > 0$, all i, r .

$$\hat{x}_{iN} = \hat{y}_{iN} = \frac{c_N}{a_{iN}} \quad i = 1, \dots, n; \quad \hat{v}_N = c_N; \quad \frac{1}{c_N} = \sum_{i=1}^N \frac{1}{a_{iN}}$$

and, in general, the solution is

$$(5) \quad \hat{x}_{ir} = \hat{y}_{ir} = \frac{c_r}{a_{ir} + \hat{v}_{r+1} P_{ir}} \quad i=1, \dots, n; \quad \hat{v}_r = c_r; \quad \frac{1}{c_r} = \sum_{i=1}^n \frac{1}{a_{ir} + \hat{v}_{r+1} P_{ir}}$$

For the N^{th} move, the optimal strategies are identical to those of an n -cell game with diagonal payoff matrix A_N . This is to say, P2 chooses a hiding cell with probability which is inversely proportional to the payoff for that cell. For moves other than the N^{th} , an optimal strategy depends on the current payoff and the continuation probability as shown in equation (5). This seems to be a "reasonable" optimal strategy for P2. On move N , the theory tells P1 to look with the highest probability in the cell with the smallest probability of detection because, in a sense, P2 is likely to hide in the cell with the lowest probability of detection.

We close the discussion of the finite sequential game with a simple example. Consider a game with two cells and the same A and P matrix for every move. This game will have, at most, three moves. Later, we compare this game with one where an infinite number of moves is allowed. The given information for the game is

$$A = \begin{bmatrix} .1 & .2 \\ .3 & .1 \end{bmatrix} \quad P = \begin{bmatrix} .8 & .7 \\ .6 & .7 \end{bmatrix} \quad N = 3$$

We solve this game by a simple graphical method. Starting with

$\hat{v}_4 = 0$ in equation (3), we seek to find

$$\hat{v}_3 = \max_X \min_Y X_3^t A Y_3$$

We find

$$\hat{v}_3 = \frac{1}{6}, \quad \hat{X}_3 = \left(\frac{2}{3}, \frac{1}{3} \right)$$

Using $\hat{v}_3 = \frac{1}{6}$, we must now solve the game with payoff matrix

$$A + \hat{v}_3 P = \begin{bmatrix} 1.4 & 1.9 \\ 2.4 & 1.3 \end{bmatrix}$$

The solution is

$$\hat{v}_2 = .286, \quad \hat{X}_2 = \left(\frac{11}{16}, \frac{5}{16} \right)$$

and to obtain the value of the sequential game and an optimal strategy

for P1 on move 1, we solve the game with payoff matrix

$$A + \hat{v}_2 P = \begin{bmatrix} .328 & .400 \\ .417 & .300 \end{bmatrix}$$

We obtain the solution

$$\hat{v}_1 = .362, \quad \hat{X}_1 = \left(\frac{8}{13}, \frac{5}{13} \right)$$

As we have shown, \hat{v}_1 is the value of the sequential game and \hat{X}_r ,

$r = 1, 2, 3$, is an optimal strategy for P1 on move r .¹

¹We could also have readily obtained optimal strategies for P2.

3.5 Formulation of the Infinite Sequential Game

In this section, we allow an infinite number of moves in the sequential game. Before giving an analytic formulation, we discuss some of the features of the game. In the infinite sequential game there is no maximum number of moves. The continuation probabilities alone control the termination of the game. To obtain a manageable analytic problem, we must assume that one payoff matrix and one continuation probability matrix are specified for all moves. We further assume that the probability of continuing until the next move is strictly less than one for all pairs of strategies. This assumption guarantees boundedness of the expected accumulated payments received by P1; and it guarantees that the game terminates with probability one, although the number of moves may not be bounded. We will discuss these assumptions in more detail, when we consider a more complicated version of this game, in the next chapter. Now we turn to a formal definition of the game under consideration.

We assume that a search region is specified and that it is divided into n cells. If P1 chooses cell i ($i = 1, \dots, n$) and P2 chooses cell j ($j = 1, \dots, n$) on move r ($r = 1, 2, \dots$) then P1 receives from P2 the payoff

$$a_{ij}$$

and the game continues until move $r + 1$ with probability

$$(6) \quad 0 \leq p_{ij} < 1$$

Let P be the $n \times n$ matrix $P = (p_{ij})$ and A the $n \times n$ matrix $A = (a_{ij})$. A is the payoff matrix, and P is the matrix of continuation probabilities for every move. We further assume that the game is zero sum and that P_1 is the maximizing player.

The game which we have defined above is one of "perfect recall"; and by Kuhn's [1] theorem, a "behavior strategy" is optimal. Briefly, a behavior strategy is defined with reference to the information sets in the game. If a player uses a behavior strategy, he plays the same mixed strategy over the alternatives in an information set each time the information set is reached, regardless of the past history of the game. In the infinite sequential game, there is only one information set and, therefore, a behavior strategy is simply a mixed strategy which is used for every move. We restrict our attention to these strategies.

Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ be behavior strategies (mixed strategies over the alternatives) for P_1 and P_2 respectively. For example, P_1 chooses alternative i with probability x_i on every move. The expected accumulated payment received by P_1 , $v(X, Y)$, is simply the sum over all r of the probability that the game lasts until move r times the payment to P_1 for move r ,

$$(7) \quad v(X, Y) = \sum_{r=0}^{\infty} (X^t P Y)^r X^t A Y$$

The above sum converges, since (6) implies $0 \leq X^t P Y < 1$ for all strategies X, Y . For convenience, we define the matrix $Q = (q_{ij})$ with $q_{ij} = 1 - p_{ij}$ all i, j . Then Q is the matrix of positive termination probabilities. Equation (7) may be written as

$$(8) \quad v(X, Y) = \frac{X^t A Y}{1 - X^t P Y} = \frac{X^t A Y}{X^t Q Y}$$

von Neumann [3] first established the existence of a unique value v and optimal strategies \hat{X} and \hat{Y} for the form in (8), i. e., there exists a unique real number v and strategies \hat{X}, \hat{Y} such that

$$(9) \quad \frac{X^t A \hat{Y}}{X^t Q \hat{Y}} \leq v \leq \frac{\hat{X}^t A Y}{\hat{X}^t Q Y} \quad \text{all strategies } X, Y$$

An elementary proof of this fact was subsequently given by Loomis [1], and this result is a special case of Shapley's [1] more general "stochastic game". Neuts [1] formulated and solved a special case of the infinite sequential game. His P matrix was a diagonal matrix and his A matrix also had a special form.

3.6 Solution by Linear Programming Methods

In the last section, we formulated the game of interest and noted the existence of a solution (a value and optimal strategies). However, there are no known methods for computing a solution. In this section, we develop a computational method to approximate a solution. The method is based on a linear programming formulation of the game with an unknown parameter in the constraints. We show that this parameter is equal to the value of the game if, and only if, the optimal objective function of the linear program is zero. The remainder of our discussion is then devoted to a method for approximating the required value of the parameter.

To begin, we establish Lemma 1 which relates the solution of the infinite sequential game to the solution of an ordinary two-person zero-sum game.

Lemma 1 A necessary and sufficient condition for v to be the value of the infinite sequential game and \hat{X} , \hat{Y} optimal strategies is that the two-person zero-sum game with payoff matrix $A - vQ$ has value zero and optimal strategies \hat{X} , \hat{Y} .

Proof For the game $A - vQ$ to have value zero and optimal strategies \hat{X} , \hat{Y} , it is necessary and sufficient that

$$(10) \quad X^t(A - vQ)\hat{Y} \leq 0 \leq \hat{X}^t(A - vQ)Y \quad \text{all strategies } X, Y$$

But, $X^tQY > 0$ for all strategies, X, Y . Hence, v, \hat{X}, \hat{Y} satisfy (10)

if, and only if,

$$(10a) \quad \frac{\hat{X}^t A \hat{Y}}{\hat{X}^t Q \hat{Y}} \leq v \leq \frac{\hat{X}^t A Y}{\hat{X}^t Q Y} \quad \text{all strategies } X, Y.$$

Equation (10a) is a necessary and sufficient condition for v to be the value of the infinite sequential game and \hat{X} , \hat{Y} optimal strategies. Hence, the lemma is true.

Lemma 1 immediately suggests a method for computing v . The main idea is to choose a number s and compute the value of the game $A - sQ$. If the value of $A - sQ$ is zero, then $s = v$ and we are finished. If the value of $A - sQ$ is not zero, then we want to choose a new value of s , say s_1 , such that the value of $A - s_1Q$ is "closer" to zero than the value of $A - sQ$. We begin by formulating the game $A - sQ$ as a linear program.

Consider the linear program

$$(11) \quad \begin{aligned} &\text{Max } u_s \\ &u_s e^t - X_s^t (A - sQ) \leq 0 \\ &X_s^t e = 1 \\ &X_s \geq 0 \end{aligned}$$

where e is the $n \times 1$ vector of all "ones", X_s is an $n \times 1$ vector, and s is a fixed scalar. Let \hat{u}_s , \hat{X}_s be an optimal solution to (11). Then from Charnes [1], \hat{X}_s is an optimal strategy for P1 and \hat{u}_s is the value of the game $A - sP$, (s fixed). Of course, an optimal strategy \hat{Y}_s for

P2 is part of an optimal solution to the dual of (11), and \hat{Y}_s is available when (11) is solved by the simplex method.

Next, we examine the variation in \hat{u}_s which results from a change in s . We consider a perturbation from s to $s + \xi$ in problem (11), and we want to relate \hat{u}_s to $\hat{u}_{s+\xi}$. Accordingly, we add and subtract the vector $\xi X_s^t Q$ from the constraints of (11) and obtain the following equivalent linear program

$$\begin{aligned}
 & \text{Max } u_s \\
 & u_s^t e^t - X_s^t (A - (s + \xi) Q) - \xi X_s^t Q \leq 0 \\
 (12) \quad & X_s^t e = 1 \\
 & X_s \geq 0
 \end{aligned}$$

We seek to obtain a linear programming formulation of the game

$A - (s + \xi) Q$ from (12). Hence, we let $\bar{q} = \max_{i,j} q_{ij}$, $\bar{q} = \min_{i,j} q_{ij}$

and then for $\xi > 0$

$$(13) \quad \xi \bar{q} e^t \leq \xi X_s^t Q \leq \xi \bar{q} e^t \quad \text{all strategies } X_s$$

Now consider the following linear program

$$\begin{aligned}
 & \text{Max } u' \\
 & u'^t e^t - X'^t (A - (s + \xi) Q) \leq \xi \bar{q} e^t \\
 (14) \quad & X'^t e = 1 \\
 & X' \geq 0
 \end{aligned}$$

Problem (14) is "less constrained" than (12). Therefore, the

respective optimal solutions must satisfy ("hats" on the variables denote optimal values)

$$(15) \quad \hat{u}' \geq \hat{u}_s$$

Notice that the right-hand side of the constraints in (14) is a constant vector. We bring this vector over to the left-hand side of the constraints and make the change of variable

$$(16) \quad u_{s+\xi} = u' - \xi \bar{q}$$

to obtain the program

$$(17) \quad \begin{aligned} \text{Max } (u_{s+\xi} + \xi q) \\ u_{s+\xi} e^t - X^t (A - (s + \xi) Q) &\leq 0 \\ X^t e &= 1 \\ X &\geq 0 \end{aligned}$$

But, (17) is the desired linear programming formulation of the game $A - (s + \xi) Q$ except for the additive constant $+\xi q$ in the objective function. Hence,

$$\hat{u}' = \hat{u}_{s+\xi} + \xi \bar{q}$$

and from (15) and the above equation

$$\hat{u}_{s+\xi} + \xi \bar{q} \geq \hat{u}_s$$

By using the left-hand side of (13), we get by a similar argument

$$\hat{u}_{s+\xi} + \xi \bar{q} \leq \hat{u}_s$$

Thus for $\xi > 0$

$$(18) \quad \hat{u}_s - \xi \bar{q} \leq \hat{u}_{s+\xi} \leq \hat{u}_s - \xi \bar{q}, \quad \xi > 0$$

and for $\xi < 0$ we can derive the relationship

$$(19) \quad \hat{u}_s - \xi \bar{q} \leq \hat{u}_{s+\xi} \leq \hat{u}_s - \xi \bar{q}, \quad \xi < 0$$

Equations (18) and (19) give the desired relationships. We can choose a starting value of s and then subsequently perturb \hat{u}_s towards zero.

Before giving a tactical example, we determine two numbers m and M ($m \leq M$) such that $\hat{u}_m \geq 0$ and $\hat{u}_M \leq 0$. Then since \hat{u}_s is a continuous function of s ,¹ $\hat{u}_s = 0$ for some s in the range $m \leq s \leq M$.

Suppose we choose

$$(20) \quad m = \min_{i,j} \frac{a_{ij}}{q_{ij}}, \quad M = \max_{i,j} \frac{a_{ij}}{q_{ij}}$$

then

$$m q_{ij} \leq a_{ij}, \quad M q_{ij} \geq a_{ij} \quad \text{all } i, j$$

From the constraints of (11), we see that

$$\hat{u}_s = \min_j \sum_{i=1}^n \hat{x}_i (a_{ij} - s q_{ij})$$

thus

$$\hat{u}_m \geq 0, \quad \hat{u}_M \leq 0$$

With certain restrictions on the elements a_{ij} , we can derive tighter bounds than m and M ; but for our purposes, the bounds given here are

¹This fact is clear from the foregoing derivation.

adequate. Since the value of the game v satisfies $\hat{u}_v = 0$, v must be in the range $m \leq v \leq M$ and we restrict our attention to this range.

3.7 Tactical Payoffs and an Example

Two reasonable objectives for P1 in tactical situations are:

1. Minimize the time to detect P2.
2. Maximize the probability of detecting P2.

Of course, P2 maximizes when P1 minimizes and vice versa.

The first objective is discussed in the next section and the resulting game can be formulated as a linear program. No iterative solution technique is required. To obtain objective 2 above, we interpret the a_{ij} as the probability that P1 detects P2 in one move. We also allow the game to terminate by several methods. For example, the game terminates if P1 detects P2 or P2 sinks P1 or P2 escapes from the search region. Hence, the probability that the game continues until the next move is no larger than one minus the probability that P1 detects P2, i. e., $p_{ij} \leq 1 - a_{ij}$ all (i, j) . With this condition, the value v will satisfy $0 \leq v \leq 1$, and v is in fact the probability that P1 eventually detects P2. Also, notice if we require that P1 either detects P2 or the game continues, then $p_{ij} = 1 - a_{ij}$ (all i, j). From equation (8) $v = 1$, i. e., P1 eventually detects P2 with probability one and all strategies are optimal.

To illustrate the method developed in the last section, we present the following example. We assume that the search region consists of two cells and that P1 wants to maximize the probability of detecting P2.

The following payoffs A and stop probabilities Q are given.

$$A = \begin{bmatrix} .1 & .2 \\ .3 & .1 \end{bmatrix} \quad Q = \begin{bmatrix} .2 & .3 \\ .4 & .3 \end{bmatrix}$$

With the above data, linear program (11) becomes

$$\begin{aligned} &\text{Max } u_s \\ &u_s - x_{s1} (.1 - .2s) - x_{s2} (.3 - .4s) \leq 0 \\ (21) \quad &u_s - x_{s1} (.2 - .3s) - x_{s2} (.1 - .3s) \leq 0 \\ &x_{s1} + x_{s2} = 1 \\ &x_{s1}, x_{s2} \geq 0 \end{aligned}$$

To apply our method, we need an initial value of s . The bounds from equation (20) for this example are

$$\frac{1}{3} = m \leq v \leq M = \frac{3}{4}$$

We choose our initial value of s between the above bounds; and for convenience, we try $s = \frac{1}{2}$. The resulting optimal solution to (21) is

$$\hat{u}_{\frac{1}{2}} = 0.025, \quad \hat{x}_1 = \frac{3}{4}, \quad \hat{x}_2 = \frac{1}{4}$$

Now we want to choose s to get $\hat{u}_s \leq 0$. From the discussion following equation (20), we have $\hat{u}_M \leq 0$; and for convenience, we select $s = .7$.

The resulting optimal solution to (21) is

$$\hat{u}_{.7} = -0.288, \quad \hat{x}_1 = \frac{13}{16}, \quad \hat{x}_2 = \frac{3}{16}$$

We use linear interpolation between $s = .5$ and $s = .7$ to approximate the value of s which gives $\hat{u}_s = 0$, i. e.,

$$s = .5 + (.2) \frac{100}{215} = 0.593$$

Now $\hat{u}_{.593} \cong 0$, we conclude that $v \cong .593$. We round v off to $.6$ and solve problem (21) with $s = .6$ to obtain the following optimal strategies

$$\hat{x}_1 = \frac{7}{9}, \quad \hat{x}_2 = \frac{2}{9}$$

$$\hat{y}_1 = \frac{5}{9}, \quad \hat{y}_2 = \frac{4}{9}$$

In this example, P1 can detect P2 with probability at least $v = .593$ by playing optimally. Of course, P2 can prevent P1 from obtaining a larger probability of detection than $.593$ by also playing optimally.

We may compare this solution to the solution of the finite game with the same payoff matrix and stop probability matrix. (See section 3.4.) The probability that P1 detects P2 in at most three steps was $v_1 = .302$. Also, when the game lasts one step, we have an n -cell game of Chapter 2. In this case, the probability of detection is $.167$. This completes the discussion of the example, and we turn to a special case of the most general game.

3.8 A Special Case: Minimax the Expected Duration of the Game

We investigate the special case when P1 seeks to minimize the expected duration of the game and P2 seeks to maximize it. To obtain this objective, we must take $a_{ij} = 1$ (all i, j). Then, from equation (7), the expected accumulated payment received by P1 is the expected duration of the game. With all $a_{ij} = 1$, equation (8) becomes

$$v(X, Y) = \frac{1}{X^t Q Y}$$

We want P1 to be the minimizing player so we seek to solve the equation

$$(22) \quad v = \min_X \max_Y \frac{1}{X^t Q Y} \equiv \frac{1}{\hat{X}^t Q \hat{Y}}$$

Clearly, v , \hat{X} , and \hat{Y} satisfy (22) if, and only if, they satisfy

$$(23) \quad \frac{1}{v} = \max_X \min_Y X^t Q Y = \hat{X}^t Q \hat{Y}$$

Hence, P1 can minimax the expected duration of the game by maximizing the probability that the game terminates in one step.

We can solve equation (23) by the following familiar linear programming formulation of a matrix game

$$(24) \quad \begin{aligned} &\text{Max } u \\ &u e^t - X^t Q \leq 0 \\ &X^t e = 1 \\ &X \geq 0 \end{aligned}$$

Let \hat{u} , \hat{X} be an optimal solution to (24) and \hat{Y} a dual optimal strategy.

Then

$$\hat{u} = \hat{X}^t Q \hat{Y} = \max_X \min_Y X^t Q Y$$

and from the equivalence of the optimal solutions to (22) and (23)

$$\frac{1}{\hat{u}} = \min_X \max_Y \frac{1}{X^t Q Y} = \frac{1}{\hat{X}^t Q \hat{Y}}$$

This is the desired solution to the infinite sequential game when the objective is to minimax the expected duration of the game.

We solve the example of the last section when minimax the expected duration is desired. Here

$$Q = \begin{bmatrix} .2 & .3 \\ .4 & .3 \end{bmatrix}$$

and problem (24) becomes

Max u

$$u - x_1 (.2) - x_2 (.4) \leq 0$$

$$u - x_1 (.3) - x_2 (.3) \leq 0$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

The solution to this simple linear program is

$$\hat{u} = .3, \quad \hat{x}_1 = \frac{1}{2}, \quad \hat{x}_2 = \frac{1}{2}$$

Thus, the minimax expected duration is $\frac{1}{\hat{u}} = \frac{10}{3}$ moves. And,

$\hat{X} = (\hat{x}_1, \hat{x}_2)$ is an optimal strategy for P1.

3.9 A Stop Strategy and Dominance

A feature that can easily be included in the sequential games is a stop strategy for P1, which will allow P1 to terminate the search if he so chooses. In the finite game, this option can be included by simply adjoining an additional row ($n + 1$) to each of the matrices A_r and P_r . Since row $n + 1$ is to be a stop strategy, we require that row $n + 1$ of the matrix P_r contains all zeros. By solving this new game, we find the moves for which P1 chooses row $n + 1$ with positive probability or zero probability. In this way, we will have an optimal stopping rule for the game. We do not pursue this point further, since the moves for which P1 employs his stop strategy with probability zero will depend on the specified data (A_r and P_r).

We also apply the idea of a stop strategy to the infinite sequential game. Again, we adjoin an additional row $n + 1$ to the A and P matrix, with row $n + 1$ of the P matrix all zeros. We could establish a sufficient condition for P1 to choose row $n + 1$ with probability zero. In this case, P1 will allow the game to terminate by the already specified means and P1 will not abandon the search at any move. To obtain this sufficient condition, we would require the notion of dominance for the infinite sequential game. Dominance in the sequential game is equivalent to ordinary dominance in the equivalent two-person zero-sum game with payoff matrix $A - vQ$. The desired result easily follows from this notion of dominance.

CHAPTER IV - TACTICAL STOCHASTIC GAMES¹

4.1 Introduction

(a) The Problem

This chapter is concerned with the development of models and methods for finding optimal tactics in an idealization of Antisubmarine Warfare (ASW). We view the ASW problem as a game of pursuit between the hunter-killer force (player 1) and a possible submarine (player 2). The pursuit begins with a contact which is an indication of a possible submarine by the sensors of one or more units of the hunter-killer force. The pursuit ends when the contact is "caught" or, in some cases, evades the hunter-killer force. A catch may correspond to the attainment of one of several military objectives such as positive identification that the contact is or is not a submarine or sinking of the submarine. In any event, a catch is a specified terminal condition for the pursuit.

The status of the pursuit at every move t ($t = 1, 2, \dots$) is taken to be one of a finite number of possible states. A state summarizes the tactical information which is available to both players for decision

¹Much of the work in this chapter is also contained in Charnes and Schroeder [1].

making. For example, each state may correspond to one of a finite number of possible configurations of the hunter-killer forces which may hold the contact on their sensors. Then, at every move t , each player determines the state of the pursuit by observing the configuration of the hunter-killer forces which are holding the contact. Thus, a finite collection of states numbered $i = 1, \dots, n$ is specified. When the pursuit has not terminated, it must be in one and only one of these states at each move.

The structure of the problem also includes a finite collection of tactical plans (decisions) associated with each state. A plan specifies the tactics which a player will use until his next move. In the most general case, we assume that the players simultaneously choose a plan after the state of the pursuit is observed. When the pursuit is in state i , we number the available plans $k = 1, \dots, M_i$ and $h = 1, \dots, N_i$ for players 1 and 2 respectively. When the players move, they each choose a plan and thereby jointly determine an immediate "payoff" from player 2 to player 1 and a transition probability distribution over the states. Before the next move is made, the game transits to one of the states or terminates according to the chosen probability distribution.¹ We assume that the game is zero sum.

¹Each move which we consider consists of both a personal and chance move in the sense of von Neumann and Morgenstern [1].

We consider different payoffs corresponding to different ASW objectives. Two reasonable ASW objectives for player 1 are:

- (1) Minimize the expected duration of the game.
- (2) Maximize the probability of a catch.

In case (1), the payoff for every pair of state and plan is the time taken by one move. Or, the hunter-killer force wishes to catch the submarine in minimum time. With objective (2), we must have at least two terminal conditions for the pursuit. For in this case, the hunter-killer force attempts to maximize the probability of catching the submarine and is faced with the possibility that the pursuit may terminate with conditions other than a catch.

In short, the problem consists of a finite collection of states which summarizes the tactical information available to both players. At each move, the players observe the state of the game and each player chooses a tactical plan from a finite collection. The chosen tactical plans jointly determine an immediate payoff and a transition probability distribution over the states. Before the next move is made, the game transits to one of the states or terminates according to the chosen probability distribution. Our task is to find an optimal strategy for each player. A strategy is a decision (possibly randomized) for each state and move. An optimal strategy is one of a minimax pair for the total expected payoff. For convenience, unless otherwise noted, we shall take

player 1 to be the maximizing, and player 2 the minimizing, player.

(b) The Models

To describe the above ASW situation, we consider a basic model and four variants. The basic model is a stochastic game due to Shapley [1]. We call this game a Terminating Stochastic Game (TSG) to distinguish it from the non-terminating variant introduced by Hoffman and Karp [1]. Shapley defined a vector value for a TGS and employed an ingenious argument to establish its existence and that of optimal strategies. The methods and representations he employed were of a nonlinear character. We show, however, that linear programming can be used to characterize the value of the game and its optimal strategies as well as to obtain them to within a desired degree of approximation. In addition, we determine the effect of near-optimal strategies on the total expected payoff for the TSG.

Next, we discuss two variants of the TSG which lend considerably more realism to the game for ASW purposes. The first involves a modified assumption on the transition probabilities from that employed by Shapley. No change in the solution techniques developed for the basic game is required by this modification although it enlarges the class of problems which may be solved. The second variant involves an extension of the notion of a constrained game, Charnes, to stochastic games and is exemplified in a particular type of "constrained" TGS.

Here an implicit restriction on the duration of the game is rendered by means of constraints on the strategies.

Another description of the ASW situation may be obtained from a TSG with perfect information. We discuss its advantages in describing the ASW problem and exhibit a linear program whose solution yields the value and optimal strategies for a general TSG with perfect information. The existence and uniqueness of the value is also established directly from this linear program.

Finally, we introduce a finite version of a TSG. This finite TSG is applicable to the ASW situation when the pursuit is known to terminate in, at most, a finite number of steps. This finiteness allows us to relax certain assumptions which are required in the infinite case and, thus, additional realism can be introduced into the model. Again, however, our basic linear programming techniques hold good and yield constructive procedures.

4.2 Formulation of a Terminating Stochastic Game

In this section, we define the TSG and present two basic theorems due to Shapley. A TSG is played in a sequence of moves. At each move, the game is said to be in one of a finite number of states numbered $i = 1, \dots, n$. If the game is in state i ($i = 1, \dots, n$) and player 1, the maximizing player, chooses alternative k and player 2 chooses alternative h , then the payoff to player 1 from player 2 is

$$a_i^{kh} \quad \begin{array}{l} k = 1, \dots, M_i \\ h = 1, \dots, N_i \end{array}$$

Since we have assumed the game is zero sum, player 2 receives, of course, $-a_i^{kh}$. The choice of alternatives k and h also determines the transition probabilities:

$$\begin{array}{l} j = 1, \dots, n \\ p_{ij}^{kh} \geq 0 \quad \begin{array}{l} k = 1, \dots, M_i \\ h = 1, \dots, N_i \end{array} \end{array}$$

where p_{ij}^{kh} is the conditional probability that the game will be in state j on the next move given that it is in state i , and that strategies k and h are chosen by players 1 and 2 respectively. Hereafter, if the range of the subscripts i, j, k, h is omitted, their full range is intended.

We assume:

$$(1) \quad (i) \quad \sum_{j=1}^n p_{ij}^{kh} < 1 \quad \text{all } k, h, i$$

$$(2) \quad (ii) \quad |a_i^{kh}| < M \quad \text{all } k, h, i$$

Under these assumptions, the game terminates with probability one and the accumulated payments received by either player are bounded. To

verify this statement we let $s_i^{kh} = 1 - \sum_{j=1}^n p_{ij}^{kh} > 0$, s_i^{kh} is the

positive probability of termination given state i and decisions k and h .

Let

$$(3) \quad s = \min_{i, k, h} s_i^{kh}$$

Now the probability that the game does not terminate in N moves is not more than $(1 - s)^N$. Since this quantity tends to zero as N increases without limit, the game terminates with probability one. The accumulated payments received by either player are bounded by

$$M + (1 - s)M + (1 - s)^2 M + \dots = \frac{M}{s}.$$

A strategy for a move could depend on the entire previous history of the game play. Fortunately, it is only necessary to consider "behavior strategies" (stationary strategies), since the optimal strategies are found in this class, Kuhn [1], Shapley [1].

Def. A behavior strategy X for player 1 is an n -tuple of probability distributions $X = (X_1, \dots, X_n)$, each $X_i = (x_i^1, \dots, x_i^{M_i})$.

A similar definition holds for player 2.

If player 1 uses a behavior strategy X , he chooses the mixed strategy X_i whenever the game is in state i regardless of what move it is or of the manner of arrival at state i .

By choosing a starting state i , we obtain an infinite¹ game G_i ($i = 1, \dots, n$). A TSG, G , is defined as the collection of games $G \equiv (G_1, \dots, G_n)$. Let \hat{w}_i denote the value of G_i , the minimax of its total expected payoffs. Now we define the value of G to be the vector $\hat{W} \equiv (\hat{w}_1, \dots, \hat{w}_n)$.

We introduce a two-person zero-sum game with payoff matrix $A_i(v)$ where $A_i(v)$, $i = 1, \dots, n$, is the $M_i \times N_i$ matrix whose k - h th element is

$$(4) \quad a_i^{kh} = \sum_{j=1}^n p_{ij}^{kh} v_j$$

and v is the n -vector of real numbers $v = (v_1, \dots, v_n)$. Finally, let $\text{Val}[B]$ denote the minimax value of the two-person zero-sum game with payoff matrix B and let $X[B]$ and $Y[B]$ denote the sets of optimal mixed strategies for players 1 and 2 respectively. Now we state two basic theorems due to Shapley [1].

¹The number of moves may not be bounded.

Theorem 1 (Shapley): The value of the terminating stochastic game G is the unique solution \hat{W} of the nonlinear system of equations

$$(5) \quad \hat{w}_i = \text{Val} [A_i(\hat{W})] \quad i = 1, \dots, n$$

Theorem 2 (Shapley): The behavior strategies \hat{X}, \hat{Y} , where $\hat{X}_i \in X_i[A_i(\hat{W})], \hat{Y}_i \in Y_i[A_i(\hat{W})] \quad i = 1, \dots, n$, are optimal for the first and second players respectively in every game G_i belonging to G . These theorems provide a basis for the results of the following section.

4.3 Solution of a TSG

In this section, we develop an iterative technique which employs a contraction mapping whose unique fixed point is the value of the game. This mapping is applied recursively from a selected starting point, and each iteration of the mapping is obtained by solving a set of linear programs. Truncation of the recursive technique yields near-optimal strategies of the TSG, and we can determine in advance the effect on the total expected payoffs when such strategies are to be used.

In order to define the contraction mapping, consider the n -dimensional real vector space R^n with the norm

$$\| \gamma \| = \max_{1 \leq i \leq n} \gamma_i ; \quad \gamma = (\gamma_1, \dots, \gamma_i, \dots, \gamma_n) \in R^n$$

Let T be the mapping from R^n into R^n defined by

$$(6) \quad Tv = \beta \text{ where } \beta_i = \text{Val}[A_i(v)] \quad i = 1, \dots, n$$

In the proof of theorem 1, Shapley shows that

$$(7) \quad \|Tv_2 - Tv_1\| \leq (1 - s) \|v_2 - v_1\| \quad \text{all } v_1, v_2 \in R^n$$

where $s > 0$ is given by equation (3). Since $0 \leq 1 - s < 1$, T is a contraction mapping and, therefore, has a unique fixed point (Kolmogorov and Fomin [1]). Theorem 1 asserts that \hat{W} , the value of G , is the unique fixed point of T .

Next, we consider the sequence $\{v(t)\}$ which is defined recursively for given $v(0)$ by

$$(8) \quad v(t+1) = Tv(t), \quad t = 0, 1, \dots$$

Then, by the definition of T

$$(9) \quad v_i(t+1) = \text{Val} [A_i(v(t))], \quad \begin{array}{l} i = 1, \dots, n \\ t = 0, 1, \dots \end{array}$$

By the contraction property of T , the sequence $\{v(t)\}$ converges to \hat{W} for every fixed $v(0)$. (See Kolmogorov and Fomin [1].) Note that if we choose $v(0) = 0$, then $v(N)$ is the value of the TSG which is truncated (stopped) after N moves, if it lasts that long. We shall return to this point later.

Now the sequence $\{v(t)\}$ may be computed by linear programming. Indeed, the i^{th} program in the following collection is a linear programming formulation of the game $A_i(v(t))$, where $v(t)$ is known.¹

Accordingly, the optimal solution $\hat{u}_i(t)$ exists and satisfies

$$\hat{u}_i(t) = \text{Val} [A_i(v(t))], \quad i = 1, \dots, n.$$

$$\begin{array}{l} \text{L. P. } (i, v(t)) \\ \hline \end{array} \quad \begin{array}{l} i = 1, \dots, n \\ t = 0, 1, \dots \end{array}$$

$$\text{Max } u_i(t)$$

$$\text{Subject to:} \quad u_i(t) - \sum_{k=1}^{M_i} x_i^k(t) (a_i^{kh} + \sum_{j=1}^n p_{ij}^{kh} v_j(t)) \leq 0, \quad h = 1, \dots, N_i$$

¹This formulation is a variant of that in Charnes [1] which has the same advantage that the dual programs correspond precisely to the play problems of the respective players.

$$\sum_{k=1}^{M_i} x_i^k(t) = 1$$

$$x_i^k(t) \geq 0, \quad k = 1, \dots, M_i$$

Given $v(t)$, we compute $\hat{U}(t) = (\hat{u}_1(t), \dots, \hat{u}_n(t))$ from the above linear programs and set $v(t+1) = \hat{U}(t)$. In this manner, the sequence $\{v(t)\}$ for given $v(0)$ is generated.

When computing the sequence $\{v(t)\}$, it is desirable to have a stopping criterion which insures a desired approximation to \hat{W} . More precisely, given arbitrary $\epsilon > 0$, we will find an integer N such that $\|\hat{W} - v(N)\| \leq \epsilon$. Returning to the contraction mapping T and recalling that $T\hat{W} = \hat{W}$, we have from equation (7)

$$(10) \quad \|\hat{W} - T^{N+1}v\| \leq (1-s)\|\hat{W} - T^Nv\|, \quad \text{all } v \in R^n.$$

Also, by the triangle inequality

$$\|\hat{W} - T^Nv\| \leq \|\hat{W} - T^{N+1}v\| + \|T^{N+1}v - T^Nv\|$$

Thus $\|\hat{W} - T^Nv\| \leq (1-s)\|\hat{W} - T^Nv\| + \|T^{N+1}v - T^Nv\|$, and

$$(11) \quad s\|\hat{W} - T^Nv\| \leq \|T^{N+1}v - T^Nv\|, \quad \text{all } v \in R^n.$$

But, by definition

$$T^m v(0) = v(m), \quad m = 0, 1, \dots$$

Therefore

$$\|\hat{W} - v(N)\| \leq \frac{1}{s} \|v(N+1) - v(N)\|.$$

If we compute the sequence $\{v(t)\}$ until $\|v(N+1) - v(N)\| \leq s\epsilon$, then $\|\hat{W} - v(N)\| \leq \epsilon$. The actual number of iterations required will depend, in general, on $\|\hat{W} - v(0)\|$, ϵ and s .

One may also easily bound by R the maximum number of iterations required after one iteration is computed. For, observe that for every integer $m \geq 1$

$$(12) \quad s \|\hat{W} - T^m v\| \leq \|T^{m+1} v - T^m v\| \leq (1-s)^m \|Tv - v\|$$

Now, after $v(1)$ is computed from $v(0)$, choose R such that

$$\frac{(1-s)^R}{s} \|v(1) - v(0)\| \leq \epsilon$$

$$\text{Then} \quad \|\hat{W} - v(R)\| \leq \epsilon.$$

By the inequalities in (12), $R \geq N$ where N is the stopping point obtained by the methods of the preceding paragraph. We conclude that after one iteration we will have an upper bound on the total number of iterations required for given accuracy.

Next, we investigate the effect of near-optimal strategies on the total expected payoff. As before, let $\hat{u}_i(N)$ and $\hat{X}_i(N) = (\hat{x}_i^1(N), \dots, \hat{x}_i^{M_i}(N))$ be an optimal solution to L.P. (i, $v(N)$) and assume, given $\epsilon > 0$, that N is chosen such that $\|\hat{U}(N) - v(N)\| \leq s\epsilon$ where $\hat{U}(N) = (\hat{u}_1(N), \dots, \hat{u}_n(N))$. Then from equation (11), $\|\hat{U}(N) - \hat{W}\| \leq \epsilon$. Let $\hat{Y}_i(N)$ be an optimal strategy for player 2 in the game $A_i(v(N))$.

Then $\hat{Y}_i(N)$ is an optimal strategy in the dual to L. P. (i, v(N)) and is, of course, at hand when the direct problem is solved. (See Charnes [1].)

Let $U = (u_1, \dots, u_n)$ be the expected payoff in the TSG when the known strategies $\hat{X}(N)$ and $\hat{Y}(N)$ are used in every move of G. We wish to find the difference in norm between U and the value of G, \hat{W} . First, we compute the difference in norm between U and $\hat{U}(N)$. Let

$$\hat{p}_{ij} = \sum_{k, h} \hat{x}_i^k(N) p_{ij}^{kh} \hat{y}_i^h(N) \text{ and } \hat{a}_i = \sum_{k, h} \hat{x}_i^k(N) a_i^{kh} \hat{y}_i^h(N).$$

Then U is given by the solution to the system

$$(13) \quad u_i = \hat{a}_i + \sum_{j=1}^n \hat{p}_{ij} u_j, \quad i = 1, \dots, n.$$

This solution is unique since $0 \leq \hat{p}_{ij} < 1$, all i, j . Now $\hat{U}(N)$ is related to $v(N)$ by the linear programs L. P. (i, v(N)), $i = 1, \dots, n$. From primal-dual considerations

$$(14) \quad \hat{u}_i(N) - \hat{a}_i - \sum_{j=1}^n \hat{p}_{ij} v_j(N) = 0, \quad i = 1, \dots, n.$$

Subtracting equation (14) from equation (13), we obtain

$$u_i - \hat{u}_i(N) = \sum_{j=1}^n \hat{p}_{ij} (u_j - v_j(N)), \quad i = 1, \dots, n.$$

From our assumption $\|\hat{U}(N) - v(N)\| \leq s\epsilon$, we may write

$$\hat{u}_j(N) - v_j(N) = s\epsilon_j, \quad j = 1, \dots, n \text{ with } |\epsilon_j| \leq \epsilon$$

Then the desired relationship between U and $\hat{U}(N)$ is

$$u_i - \hat{u}_i(N) = \sum_{j=1}^n \hat{p}_{ij} (u_j - \hat{u}_j(N) + s \epsilon_j), \quad i = 1, \dots, n.$$

Further

$$|u_i - \hat{u}_i(N)| \leq \sum_{j=1}^n \hat{p}_{ij} |u_j - \hat{u}_j(N)| + \sum_{j=1}^n \hat{p}_{ij} s |\epsilon_j|, \quad i = 1, \dots, n$$

and since $\sum_{j=1}^n \hat{p}_{ij} \leq (1 - s)$, $i = 1, \dots, n$, we have

$$\|U - \hat{U}(N)\| \leq (1 - s) \|U - \hat{U}(N)\| + s(1 - s) \epsilon$$

$$\|U - \hat{U}(N)\| \leq (1 - s) \epsilon$$

Finally, the difference in norm between U and \hat{W} is bounded by

$$\|U - \hat{W}\| \leq \|U - \hat{U}(N)\| + \|\hat{U}(N) - \hat{W}\| \leq (1 - s) \epsilon + \epsilon = (2 - s) \epsilon.$$

From the above equation, we see that one can find a priori an integer N such that the behavior strategies $\hat{X}(N)$ and $\hat{Y}(N)$ can be used in the TSG, G ; and the total expected payoff obtained will be as close to \hat{W} as prescribed.

We summarize the results of this section with the following

Theorem 3: Let the sequence $\{v(t)\}$ be defined by equation (9)

and let \hat{W} be the value of the TSG. For given $\epsilon > 0$, define N as the smallest integer for which

$$\|v(N+1) - v(N)\| \leq s \epsilon$$

then

$$(i) \quad \|\hat{W} - v(N)\| \leq \epsilon$$

Also, let $\hat{X}_i(N)$ and $\hat{Y}_i(N)$ be optimal strategies for P1 and P2 respectively in the game $A_i(v(N))$, $i = 1, \dots, n$, and let U be the accumulated payoff received by P1 when these strategies are used in every move of the TSG, then

$$(ii) \quad \|U - \hat{W}\| \leq (2 - s) \epsilon$$

To recapitulate in part, we have defined a nonlinear contraction mapping T whose unique fixed point is \hat{W} . We have shown how to replace the fixed point problem by optimizing a linear programming formulation. In this way, the successive terms of the sequence $\{T^n v\}$ were computed and a stopping criterion was developed which insured the desired approximation to \hat{W} . Finally, the linear programs L. P. $(i, v(t))$, $(i = 1, \dots, n)$ yielded a dual pair of optimal strategies $\hat{X}(t)$, $\hat{Y}(t)$ and we obtained the effect on the total expected payoff when these strategies are used in the TSG.

4.4 Another Solution Method

In this section, we develop another iterative method to compute the value and optimal strategies for a TSG. This method is closely related to one proposed by Hoffman and Karp [1] for nonterminating stochastic games, and it is also an extension of Howard's [1] policy iteration method to stochastic games. One iteration of our method consists of starting with a strategy for P1 and, in a certain way, computing a new strategy for P1. Thus, the method iterates on strategies for P1 as opposed to the method of the last section which iterated on the "state values".

Next, we describe the method and then establish some properties of the quantities which are generated on successive iterations.

Method II

1. Choose a behavior strategy $X(0) = (X_1(0), X_2(0), \dots, X_n(0))$.
2. Given $X(t)$ with $X_i(t) = (x_i^1(t), x_i^2(t), \dots, x_i^{M_i}(t))$, $i = 1, \dots, n$, find the solution to the system of equations

$$(15) \quad w_i(t) = \min_h \sum_{k=1}^{M_i} x_i^k(t) \left[a_i^{kh} + \sum_{j=1}^n p_{ij}^{kh} w_j(t) \right]$$

(The solution $W(t) = (w_1(t), \dots, w_n(t))$ is unique and may be found by solving a linear program of the type given in section 4.8.)

3. Now $X(t+1) = (X_1(t+1), \dots, X_n(t+1))$ is determined by finding an optimal strategy for P1, $X_i(t+1)$, in the games $A_i(W(t))$, $i = 1, \dots, n$. Return to step 2.

We show that the sequence $\{X(t)\}$ converges to an optimal strategy, \hat{X} , for P1, and that the sequence $\{W(t)\}$ converges to \hat{W} , the value of the TSG. First, we establish the following

Lemma 1: Successive solutions obtained from equation (15) satisfy

$$W(t+1) \geq W(t)$$

(The inequality holds component-wise on the above vectors.)

Proof: From equation (15),

$$(15) \quad w_i(t) \leq \sum_{k=1}^{M_i} x_i^k(t) \left[a_i^{kh} + \sum_{j=1}^n p_{ij}^{kh} w_j(t) \right] \quad \begin{matrix} h = 1, \dots, N_i \\ i = 1, \dots, n \end{matrix}$$

Now, for every strategy $Y = (y^1, y^2, \dots, y^{N_i})$ we have

$$\sum_{h=1}^{N_i} y_i^h = 1, \quad y_i^h \geq 0.$$

And, we may multiply both sides of equation (16) by y_i^h and sum over h to obtain

$$(17) \quad w_i(t) \leq \sum_{h=1}^{N_i} \sum_{k=1}^{M_i} x_i^k(t) \left[a_i^{kh} + \sum_{j=1}^n p_{ij}^{kh} w_j(t) \right] y_i^h \quad i = 1, \dots, n$$

We adopt the simplified notation

$$a_i(X_i, Y_i) = \sum_{h=1}^{N_i} \sum_{k=1}^{M_i} x_i^k a_i^{kh} y_i^h \quad i = 1, \dots, n \text{ and}$$

$$a(X, Y) = (a_1(X_1, Y_1), \dots, a_n(X_n, Y_n))$$

$$P_{ij}(X_i, Y_i) = \sum_{h=1}^{N_i} \sum_{k=1}^{M_i} x_i^k p_{ij}^{kh} y_i^h \quad i, j = 1, \dots, n.$$

$P(X, Y)$ is the $n \times n$ matrix $P(X, Y) = (p_{ij}(X_i, Y_i))$. With this notation, equation (17) becomes

$$(18) \quad W(t) \leq a(X(t), Y) + P(X(t), Y) W(t) \quad \text{all strategies } Y$$

According to the proposed method, $X_i(t+1)$ is an optimal strategy for P1 in the game $A_i(W(t))$. Let $Y_i(t+1)$ be an optimal strategy for P2 in this game. Then the pair of strategies $X(t+1), Y(t+1)$ satisfy the following saddle point condition

$$(19) \quad a(X, Y(t+1)) + P(X, Y(t+1))W(t) \leq a(X(t+1), Y) + P(X(t+1), Y)W(t) \\ \text{all strategies } X, Y$$

We set $Y = Y(t+1)$ in equation (18), $X = X(t)$ in equation (19), and use (18) and (19) together to get

$$(20) \quad W(t) \leq a(X(t+1), Y) + P(X(t+1), Y) W(t) \quad \text{all strategies } Y$$

Let \bar{Y} be a strategy for P2 which yields the solution to

$$(21) \quad W(t+1) = a(X(t+1), \bar{Y}) + P(X(t+1), \bar{Y}) W(t+1)$$

We set $Y = \bar{Y}$ in (20) and subtract (21) from (20) to get

$$(22) \quad W(t) - W(t+1) \leq P(X(t+1), \bar{Y}) [W(t) - W(t+1)]$$

For notational convenience, we let $\Delta = W(t) - W(t+1)$ and $P = P(X(t+1), \bar{Y})$, then equation (22) may be written as

$$(23) \quad \Delta + \xi = P\Delta \quad \text{where } \xi \geq 0$$

$$(I - P)\Delta = -\xi$$

Since all the elements of $P = (p_{ij})$ satisfy $0 \leq p_{ij} < 1$, $(I - P)^{-1}$ exists and all its elements are non-negative. Hence,

$$(24) \quad \Delta = -(I - P)^{-1} \xi \leq 0$$

we obtain

$$W(t) \leq W(t+1)$$

The vectors $W(t)$ are in Euclidean n -space, and the sequence $\{W(t)\}$ is monotone increasing.¹ We show in section 4.8 that the solution $W(t)$ to equation (15) is bounded from above for all t . Hence, the sequence $\{W(t)\}$ converges to a limit W^* . Now, it is clear from Method II that $W^* = (w_1^*, \dots, w_n^*)$ is the solution to

$$w_i^* = \text{Val } A_i(W^*) \quad i = 1, \dots, n$$

and, therefore, W^* is the value of the TSG (theorem 1).

We consider the sequence $\{X(t)\}$. The vectors $X(t)$ vary in a

¹One can verify that if $W(t) = W(t+1)$, then $W(t)$ is the value. Hence, $W(t) \leq W(t+1)$, with strict inequality holding for at least one component, unless the sequence has converged to its limit.

compact set, and we may extract a convergent subsequence. Let X^* be the limit of such a convergent subsequence. From section 4.8, $W(t)$ is the optimal solution to a linear programming problem. By the method used in section 3.6, we can establish that $W(t)$ is a continuous function of $X(t)$. Further, $X(t+1)$ is an optimal strategy for P1 in the game $A_i(W(t))$. From this fact and continuity, we may assert that X_i^* is an optimal strategy in the game $A_i(W^*)$. Then, from theorem 2, $X^* = (X_1^*, \dots, X_n^*)$ is an optimal strategy for P1 in the TSG. We sum up with the following

Theorem 4: The sequence $\{W(t)\}$ converges to the value of the TSG and the sequence $\{X(t)\}$ converges to an optimal strategy for P1.

This completes our discussion of two methods to approximate the value and optimal strategies for a TSG. In the remainder of this chapter, we investigate extensions and special cases.

4 5 A Modified Assumption

Throughout the discussion on the TSG, we have been using the assumption

$$(i) \quad \sum_{j=1}^n p_{ij}^{kh} < 1, \quad \text{all } i, k, h$$

In this section, we consider a slightly weaker assumption¹ than (i); all other definitions and assumptions remain unchanged. For convenience, we shall change the notation for transition probabilities from

p_{ij}^{kh} to q_{ij}^{kh} . We allow

$$(25) \quad \sum_{j=1}^n q_{ij}^{kh} \leq 1 \quad (\text{equality may hold for some or all } i, k, h)$$

Thus, we permit a zero probability of termination before the next move when the game is in state i and alternatives k and h are chosen by players 1 and 2 respectively. However, we impose the following regularity condition on the q_{ij}^{kh} .

Assumption A: For all behavior strategies X and Y for players 1 and 2 respectively, the game terminates with probability one in a finite number of moves from every state i ($i = 1, \dots, n$).

Now, if assumption (i) is satisfied, then assumption A is trivially

¹ See also Denardo [1] and Derman [1]. They employ this weaker assumption for a "terminating Markovian decision process".

satisfied.

Assumption A asserts that it is possible for every behavior strategy to find a finite sequence of states leading from every state i to termination of the game or, to put it another way, the states $i = 1, \dots, n$ are transient for every behavior strategy. Let ${}^r q_{ij}^{kh}$ be the r -step transition probability from state i to state j when decisions k and h are chosen for state i and an arbitrary behavior strategy is used for states other than i . Assumption A guarantees that there exists an integer N such that

$$\sum_{j=1}^n N q_{ij}^{kh} < 1, \quad \text{all } i, k, h$$

Thus, T^N is a contraction mapping and T has a unique fixed point. (See Kolmogorov and Fomin [1].) It follows that theorems 1 and 2 are true, and all of our results of section 4.3 are valid for transition probabilities satisfying equation (25) and assumption A.

4.6 Interpretation of payoffs in ASW

To place these developments in context, let us return to the ASW situation with the aforementioned objectives: (1) minimax the expected duration of the game, and (2) maximin the probability of a catch. We now seek to exhibit appropriate numerical values for the a_i^{kh} which will encompass these two objectives.

Consider first objective (1) and assume that player 1 is the minimizing player. Suppose that the fixed behavior strategies $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ are used by players 1 and 2 respectively in G . These fixed strategies X and Y define an absorbing Markov chain with transition probabilities,

$$(26) \quad p_{ij}(X_i, Y_j) = \sum_{k=1}^{M_i} \sum_{h=1}^{N_j} x_i^k p_{ij}^{kh} y_j^h, \quad i, j = 1, \dots, n$$

and the probability of absorption in one move given state i is

$$1 - \sum_{j=1}^n p_{ij}(X_i, Y_j) > 0, \quad i = 1, \dots, n$$

Now let $w_i(X, Y)$ be the expected duration of G_i when X and Y are used in G . Then the following relationship obtains.¹

¹See Parzen [1]; Pr denotes "probability".

$$w_i(X, Y) = 1 \cdot \Pr \{ \text{terminate in one move / state } i \}$$

$$+ \sum_{j=1}^n \Pr \{ \text{go to state } j / \text{state } i \} (1 + w_j(X, Y))$$

$$= 1 - \sum_{j=1}^n p_{ij}(X_i, Y_i) + \sum_{j=1}^n p_{ij}(X_i, Y_i) (1 + w_j(X, Y))$$

$$(27) \quad w_i(X, Y) = 1 + \sum_{j=1}^n p_{ij}(X_i, Y_i) w_j(X, Y), \quad i = 1, \dots, n$$

Therefore, setting

$$a_i^{kh} = 1, \quad \text{all } i, k, \text{ and } h$$

it follows from equations (4), (5), (27) that \hat{w}_i , the solution to equation (5), is the minimax expected duration of G_i . We, thus, have formulated objective 1.

To attain objective (2), a similar analysis shows that we should define (player 1 is now the maximizing player):

$$(28) \quad a_i^{kh} = \text{the probability of a catch in one move, given } i, k, \text{ and } h.$$

Then \hat{w}_i is the maximin probability of a catch for G_i . Recall that

$$s_i^{kh} = 1 - \sum_{j=1}^n p_{ij} \quad \text{is the probability of termination in one move given } i,$$

k , and h . The probability of a catch, given i , k , and h , can be no

greater than s_i^{kh} ; thus, $0 \leq a_i^{kh} \leq s_i^{kh}$. In case $a_i^{kh} = s_i^{kh}$, all i, k, h ,

then equation (5) has the trivial solution $w_i = 1$, $i = 1, \dots, n$, i.e., the submarine is caught with probability one because the game can only terminate with a catch. If $a_i^{kh} < s_i^{kh}$, then $s_i^{kh} - a_i^{kh}$ is the non-zero probability that the submarine is not caught in one move, given i, k, h , and $\hat{w}_i \leq 1$ ($i = 1, \dots, n$) strict inequality holding for at least one i . With the indicated payoffs (28), the hunter-killer force maximizes and the submarine minimizes the probability that the submarine is caught.

4.7 A Constrained TSG

This section considers a constrained TSG. See Charnes [1] for a discussion of two-person zero-sum constrained games and their reduction to linear programming problems. By a constrained game, we mean that each player's strategies are implicitly restricted to a convex set (usually polyhedral) rather than arbitrarily chosen from the unit simplex. For concreteness, suppose that player 1 is to maximize the total expected payoff subject to a constraint on the expected duration of the game. Our task is to find a restriction on P1's strategy which will guarantee that the expected duration of the game is no greater than a specified constant, $C \geq 1$. Of course, other types of constraints can also be developed by employing the method which we propose here.

As before, let $w_i(X, Y)$ be the expected duration of G_i when the fixed behavior strategies $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$, are to be used by players 1 and 2 respectively in G . Then, $w_i(X, Y)$ is the unique solution to the following system (see equation (27)).

$$w_i(X, Y) = 1 + \sum_{j=1}^n p_{ij}(X_i, Y_i) w_j(X, Y), \quad i = 1, \dots, n$$

Let

$$w_r(X, Y) = \max_{1 \leq i \leq n} w_i(X, Y)$$

Then

$$w_r(X, Y) = 1 + \sum_{j=1}^n p_{rj}(X_r, Y_r) w_j(X, Y) \leq 1 + \sum_{j=1}^n p_{rj}(X_r, Y_r) w_r(X, Y)$$

Now

$$0 \leq \sum_{j=1}^n p_{rj}(X_i, Y_i) < 1$$

thus

$$w_r(X, Y) \leq \frac{1}{1 - \sum_j p_{rj}(X_r, Y_r)}$$

Let

$$(29) \quad \bar{X}_i = \{X_i \mid \sum_{j,k} x_i^k p_{ij}^{kh} \leq 1 - \frac{1}{C}, \quad h = 1, \dots, N_i, \quad \sum_k x_i^k = 1, \quad x_i^k \geq 0,$$

$$\text{fixed } C \geq 1\} \quad i = 1, \dots, n.$$

These are the desired constraints.

We now show that $w_i(X, Y) \leq C$ for all $X_i \in \bar{X}_i$ and arbitrary strategies Y_i , ($i = 1, \dots, n$). To substantiate this claim consider an arbitrary strategy Y_i . Then for all $X_i \in \bar{X}_i$

$$\sum_{j=1}^n p_{ij}(X_i, Y_i) = \sum_{j,k,h} x_i^k p_{ij}^{kh} y_i^h \leq 1 - \frac{1}{C}, \quad i = 1, \dots, n$$

In particular

$$\sum_{j=1}^n p_{rj}(X_r, Y_r) \leq 1 - \frac{1}{C}$$

Therefore

$$w_i(X, Y) \leq w_r(X, Y) \leq \frac{1}{1 - (1 - \frac{1}{C})} = C, \quad i = 1, \dots, n.$$

Thus, player 1 can limit the expected duration of G to be no greater than C by always choosing a strategy from X_i when the game is in state i .

To solve for optimal strategies and the value with the additional restriction on the expected duration of the game, we adjoin the following constraints to L. P. $(i, v(t))$, $(i = 1, \dots, n)$.

$$(30) \quad \sum_{k,j} x_i^k(t) p_{ij}^{kh} \leq 1 - \frac{1}{C}, \quad h = 1, \dots, N_i$$

With these additional constraints, there may be no feasible solution to L. P. $(i, v(t))$ for some i . However, from the above development, infeasibility of the augmented L. P. $(i, v(t))$ for some i means that there is no behavior strategy for player 1 which satisfies the restriction on the expected duration of the game. This holds true for every $v(t)$ and will, therefore, be evident at the first iteration when $t = 0$. On the other hand, if player 1 does not have a behavior strategy satisfying the requirement on the duration of the game, then, for some i , L. P. $(i, v(0))$ will be infeasible. Summing up, the augmented L. P. $(i, v(0))$ is feasible for each i , if and only if the constrained game has a solution (a value and optimal strategies). If the solution exists, it may be found from the augmented L. P. $(i, v(t))$ and the iterative technique developed

in section 3.

A few comments on the choice of the constant C are in order.

First, we have required $C \geq 1$, equation (29). If $C < 1$, then $\bar{X}_i = \emptyset$ ($i = 1, \dots, n$) and the augmented I. P. $(i, v(t))$ is infeasible for all i .

This implies that no behavior strategy exists for player 1, which yields an expected duration less than one -- an obvious fact. Second, we may also establish an upper bound on C . By assumption (i) and equation (3)

$$\sum_{j=1}^n p_{ij}^{kh} \leq 1 - s, \quad \text{all } i, k, h$$

Thus, for every behavior strategy X ,

$$\sum_{j,k} x_i^k p_{ij}^{kh} \leq 1 - s, \quad \text{all } i, h$$

and the constraints (30) are redundant if $C \geq \frac{1}{s}$. Intuitively, this means that no behavior strategy for player 1 yields an expected duration greater than $\frac{1}{s}$. Therefore, the constraints (30) are nontrivial if C is chosen from the interval

$$1 \leq C < \frac{1}{s}.$$

4.8 A TSG with Perfect Information

We return to our idealization of ASW. In this section, the hunter-killer force knows or is willing to assume certain behavior of the submarine. More precisely, we assume that player 2 is playing some fixed behavior strategy which is known to player 1; thus, the game is effectively a one-person game. For instance, the hunter-killer force might assume that the submarine takes evasive action. Another example is the assumption that the submarine takes evasive action but is moving toward some objective. In practice, one might find optimal tactics for the hunter-killer force under various assumptions about the behavior of the submarine and then use the set of tactics for the most plausible behavior. The merits of this approach are:

1. The analysis is greatly simplified.
2. Less data is required.
3. If the submarine has the assumed behavior, the total expected payoff will be at least as great as in the two-person TSG.
4. All of the tactical information available to the hunter-killer force can be used in the state specification. (In the two-person case only the information available to both players can be used.)
5. The hunter-killer force has an optimal pure strategy.

Of course, the main disadvantage of this approach is that the hunter-killer force must have information on the behavior of the submarine or be willing to act as if it did and take the attendant risks.

Evidently, we are interested in a TSG with perfect information. Thus, we assume that player 2 uses precisely one strategy (pure or mixed) which is known to player 1. Accordingly, suppose that player 2 uses the behavior strategy $Y = (Y_1, \dots, Y_n)$. Define

$$p_{ij}^k \equiv \sum_{h=1}^{N_i} p_{ij}^{kh} y_i^h \text{ and } a_i^k \equiv \sum_{h=1}^{N_i} a_i^{kh} y_i^h. \text{ These are now the transition}$$

probabilities and payoffs for player 1 in the TSG with perfect information.

In this game with perfect information, the optimal strategies for player 1 are pure strategies (von Neumann and Morgenstern [1]). Thus,

Shapley's functional equation (5) may be rendered as

$$(31) \quad \hat{w}_i = \max_{1 \leq k \leq M_i} \left[a_i^k + \sum_{j=1}^n p_{ij}^k \hat{w}_j \right], \quad i = 1, \dots, n$$

This functional equation is one of a much larger class that has been shown by Charnes [2] to be amenable to linear programming analysis. By means of a linear program, we establish the existence and uniqueness of a solution to equation (31). In addition, the optimal pure behavior strategies and state values, \hat{w}_i , may be computed directly from the linear program.

In connection with the literature, equation (31) arises in two types of Markovian decision processes. The first may be called a terminating Markovian decision process. These processes have been

studied under the modified assumption of section 4 by Derman [1] and Eaton and Zadeh [1]. Derman obtained a linear fractional program for a terminating Markovian decision process. This linear fractional program can be reduced to a linear program by a transformation due to Charnes and Cooper [6]. The resulting linear program is precisely equivalent to the dual to problem I below. Thus, Derman's viewpoint is, in a sense, "dual" to the approach taken here. A TSG is also equivalent to a discounted Markovian decision process. For such a process, one must solve the equations

$$(31a) \quad \hat{w}_i = \max_{1 \leq k \leq M_i} \left[a_i^k + \beta \sum_{j=1}^n q_{ij}^k \hat{w}_j \right], \quad i = 1, \dots, n$$

where $0 \leq \beta < 1$ and the q_{ij}^k are transition probabilities with $\sum_{j=1}^n q_{ij}^k = 1$.

In our notation, we take $p_{ij}^k = \beta q_{ij}^k$ and we have

$$\sum_{j=1}^n p_{ij}^k = \beta \sum_{j=1}^n q_{ij}^k = \beta < 1$$

Therefore, a TSG with perfect information has precisely the same structure as a discounted Markovian decision process. For studies on discounted processes see Howard [1], Blackwell [1] and, with particular reference to equation (31a) and linear programming, see d'Epenoux [1], Balinski [2], and Denardo [1].

We employ the following linear program to solve equation (31).

Problem I

$$(32) \quad \text{Min } \sum_{i=1}^n w_i$$

$$(33) \quad \text{Subject to } w_i - \sum_{j=1}^n p_{ij}^k w_j \geq a_i^k, \quad i = 1, \dots, n, \quad k = 1, \dots, M_i$$

As may be noted, the functional of this system serves to drive the values of w_i to be the maximum over k of the right-hand side of (31).

Other functionals serving the same purpose could also be employed. The following two lemmas and theorem 3 establish that the optimal solution to problem I exists, satisfies equation (31), and is unique.

Lemma 2: An optimal solution $\hat{w} = (\hat{w}_1, \dots, \hat{w}_n)$, to problem I exists.

Proof: It is sufficient to show that problem I has a feasible solution and that its functional is bounded from below.

First, let $w_i = C$, $i = 1, \dots, n$, C is a constant. Equation (33) becomes

$$C(1 - \sum_{j=1}^n p_{ij}^k) \geq a_i^k$$

But, $1 - \sum_{j=1}^n p_{ij}^k > 0$, all i, k . Thus, we may choose C large enough to satisfy all of the above inequalities simultaneously and problem I has a feasible solution. Let $w = (w_1, \dots, w_n)$ be a feasible solution to problem I and suppose that $w_r \leq w_i$, $i = 1, \dots, n$. Then w_r must

satisfy the inequalities

$$w_r \geq a_r^k + \sum_{j=1}^n p_{rj}^k w_j \geq a_r^k + \sum_{j=1}^n p_{rj}^k w_r$$

and

$$w_r (1 - \sum_{j=1}^n p_{rj}^k) \geq a_r^k, \quad k=1, \dots, M_r$$

Let $Q = \min_{i,k} \frac{a_i^k}{1 - \sum_j p_{ij}^k}$. This minimum exists by assumptions (i) and

(ii) on page . We now have

$$w \geq w \geq \frac{a_r^k}{1 - \sum_j p_{rj}^k} \geq Q, \quad i=1, \dots, n.$$

Hence, (32) is bounded from below for every feasible solution to problem I. By boundedness and feasibility, problem I has an optimal solution.

Lemma 3: Every optimal solution to problem I satisfies equation (31).

Proof by contradiction: Let $\hat{w} = (\hat{w}_1, \dots, \hat{w}_n)$ be an optimal solution and assume for some i , say $i = r$, that

$$\hat{w}_r > a_r^k + \sum_{j=1}^n p_{rj}^k \hat{w}_j, \quad k=1, \dots, M_r.$$

Let $\hat{w}'_i = \begin{cases} \hat{w}_r - \Delta, & \text{where } \Delta > 0, i = r \\ \hat{w}_i, & i \neq r \end{cases}$

We wish to find a $\Delta > 0$ such that \hat{w}' is a feasible solution. Now

$$\begin{aligned} \hat{w}'_r - \sum_{j=1}^n p_{rj}^k \hat{w}'_j &= \hat{w}_r - \Delta - \sum_{j \neq r} p_{rj}^k \hat{w}_j - p_{rr}^k (\hat{w}_r - \Delta) \\ &= \hat{w}_r - \sum_{j=1}^n p_{rj}^k \hat{w}_j - \Delta (1 - p_{rr}^k) \end{aligned}$$

Since $1 - p_{rr}^k > 0$, we may choose $\Delta > 0$ such that

$$\hat{w}'_r - \sum_{j=1}^n p_{rj}^k \hat{w}'_j \geq a_r^k, \quad k = 1, \dots, M_r$$

For $i \neq r$

$$\begin{aligned} \hat{w}'_i - \sum_{j=1}^n p_{ij}^k \hat{w}'_j &= \hat{w}_i - \sum_{j \neq r} p_{ij}^k \hat{w}_j - p_{ir}^k (\hat{w}_r - \Delta) \\ &= \hat{w}_i - \sum_{j=1}^n p_{ij}^k \hat{w}_j + p_{ir}^k \Delta \geq a_i^k, \quad k = 1, \dots, M_i \end{aligned}$$

Therefore, \hat{w}'_i ($i = 1, \dots, n$) is a feasible solution to problem I for

some $\Delta > 0$. We also have $\sum_{i=1}^n \hat{w}'_i < \sum_{i=1}^n \hat{w}_i$. This contradicts the

assumed optimality of \hat{w} . Therefore,

$$\hat{w}_i = \max_{1 \leq k \leq M_i} \left[a_i^k + \sum_{j=1}^n p_{ij}^k \hat{w}_j \right], \quad i = 1, \dots, n.$$

We sum up with the following theorem.

Theorem 5: The optimal solution to problem I exists, it satisfies equation (31), and is unique.

Proof of uniqueness: Assume $\hat{w} = (\hat{w}_1, \dots, \hat{w}_n)$ and $w = (w_1, \dots, w_n)$ are both optimal solutions to problem I. From Lemma 3 and this assumption, there exists a set of integers $\{k(i)\}$, such that

$$(34) \quad w_i = a_i^{k(i)} + \sum_{j=1}^n p_{ij}^{k(i)} w_j, \quad i = 1, \dots, n$$

\hat{w} must be a feasible solution to problem I for the set $\{k(i)\}$, thus

$$(35) \quad \hat{w}_i \geq a_i^{k(i)} + \sum_{j=1}^n p_{ij}^{k(i)} \hat{w}_j, \quad i = 1, \dots, n.$$

Subtracting (34) from (35), we obtain

$$(36) \quad \hat{w}_i - w_i \geq \sum_{j=1}^n p_{ij}^{k(i)} (\hat{w}_j - w_j), \quad i = 1, \dots, n.$$

Let $\{\hat{k}(i)\}$ be the set of integers which gives

$$\hat{w}_i = a_i^{\hat{k}(i)} + \sum_{j=1}^n p_{ij}^{\hat{k}(i)} \hat{w}_j, \quad i = 1, \dots, n.$$

Then, we also have

$$(37) \quad \hat{w}_i - w_i \leq \sum_{j=1}^n p_{ij}^{\hat{k}(i)} (\hat{w}_j - w_j), \quad i = 1, \dots, n.$$

Consider two cases.

Case (1): Assume $\hat{w}_i - w_i < 0$ for some i ($i = 1, \dots, n$).

Then inequality (36) is not satisfied for all i ($i = 1, \dots, n$). Therefore,

$$\hat{w}_i - w_i \geq 0 \quad (i = 1, \dots, n).$$

Case (2): Assume $\hat{w}_i - w_i > 0$ for some i ($i = 1, \dots, n$).

Now inequality (37) is not satisfied for all i ($i = 1, \dots, n$). Therefore,

$\hat{w}_i = w_i = 0$ ($i = 1, \dots, n$) is the only possibility and, indeed, (36) and (37) are both satisfied when $\hat{w}_i - w_i = 0$ ($i = 1, \dots, n$). Hence, $\hat{w}_i = w_i$ ($i = 1, \dots, n$) and the theorem is true.

Since the solution to equation (31) is unique, we conclude that equation (31) is solved by problem I. Now, an optimal pure strategy for the TSG with perfect information is available from the solution to problem I or its dual. There may be more than one optimal pure strategy since the right-hand side of equation (31) may be maximized for more than one k and some i . However, alternate optima for the dual to problem I correspond to alternate optimal pure behavior strategies and vice versa. Thus, all of the optimal pure behavior strategies are available from the solution to problem I or its dual. Finally, one would normally solve for the optimal strategies from the dual to problem I since it has less constraints than problem I and, therefore, less computational effort is required.

4.9 An Example of Optimal Target Approach

We give a tactical example of a game with perfect information. Consider a situation where the searcher has a datum point of last contact, but he may or may not hold the contact on his sensors at each decision point in the pursuit. The searcher wishes to get into attack position. The states for this pursuit are determined by two observed factors, range to the datum and classification of the contact. For our purposes, range is measured in three increments, 1, 2, 3, and the classification is either a hold (H) or lost (L) contact. These two factors determine six states, 1H, 1L, 2H, 2L, 3H, 3L, where for example, 1H means the searcher is at range increment 1 from datum and is holding the contact. When the pursuit has not terminated, it must be in one of these six states. In addition, we specify two terminal states, a permanent lost state (L_0) and a successful attack state (S_0). The searcher wishes to maximize the probability of arriving at state S_0 .

Now there are four types of decisions: attack (A), decrease the range by one increment (D), increase the range by one increment (I), and stay at the present range (S). Not all of these decisions are allowed for each state. For example, the searcher cannot attack when the contact is lost (temporarily). The permissible decisions for each state and the transition probabilities are given in Figure 3.1.

FROM State	Decision	TO STATE							
		1H	1L	2H	2L	3H	3L	L _o	S _o
1H	A	.2	.3						.5
1L	S	.2	.6					.2	
	I			.3	.6			.1	
2H	D	.8	.2						
	A			.3	.4				.3
2L	D	.4	.4					.2	
	S			.3	.6			.1	
	I					.2	.8		
3H	D			.9	.1				
	A					.3	.6		.1
3L	D			.3	.6			.1	
	S					.3	.7		

Figure 3.1

TRANSITION PROBABILITIES

(Blank spaces in the table are zeros.)

We briefly review the theory of section 4.8 in order to formulate a model for this example. Recall that p_{ij}^k is the probability of transition to state j given state i and decision k . We number the non-terminal states $i = 1, \dots, 6$ and let w_i be the probability of termination in state S_0 starting from state i . From section 4.8, there is a policy (a decision for each state) which is simultaneously optimal for all starting states. Now, for each fixed policy, the probability of absorption in state S_0 is the probability of transition to S_0 in one step, plus the probability of going to some state other than S_0 and then being absorbed from there. The optimal probability of absorption is then given by

$$(40) \quad w_i = \max_k \left[p_{iS_0}^k + \sum_{j=1}^6 p_{ij}^k w_j \right] \quad i = 1, \dots, 6$$

Notice that the probability of transition to state L_0 does not appear in the above equation. Further, equation (40) is equivalent to equation (32) with $p_{iS_0}^k$ being the immediate payoff for decision k and state i . Hence, (40) may be solved by the linear program (34) and (35). Less computational effort is required to solve the dual of (34) and (35), and we exhibit this dual below.

$$(41) \quad \begin{aligned} & \text{Max} \sum_{i=1}^6 \sum_{k=1}^{M_i} p_{iS_0}^k x_{ik} \\ & \sum_{k=1}^{M_j} x_{jk} - \sum_{i=1}^6 \sum_{k=1}^{M_i} p_{ij}^k x_{ik} = 1, \quad j = 1, \dots, 6 \\ & x_{ik} \geq 0 \end{aligned}$$

In the preceding linear program, the variable x_{ik} corresponds to state i and decision k . As we have shown, the optimal positive variables x_{ik} will designate an optimal policy. (For each i , exactly one of the x_{ik} will be positive.) Also, notice that only the non-terminal states are included in (41).

The data from Figure 3.1 is arranged in the following tableau format for the linear program (41).

$P_{iS_0}^k$.5		.3					.1				
	x_{11}	x_{21}	x_{22}	x_{31}	x_{32}	x_{41}	x_{42}	x_{43}	x_{51}	x_{52}	x_{61}	x_{62}
	1-.2	-.2		-.8		-.4						= 1
	-.3	1-.6	1	-.2		-.4						= 1
			-.3	1	1-.3		-.3		-.9		-.3	= 1
			-.6		-.4	1	1-.6	1	-.1		-.6	= 1
								-.2	1	1-.3		-.3 = 1
								-.8		-.6	1	1-.7 = 1

Figure 3.2

The above linear program was solved by a standard code on the CDC 1604 computer. The resulting optimal policy and the probability of a successful attack starting from each state (dual variables) are shown next.

<u>State</u>	<u>Optimal Decision</u>	<u>Prob of a Successful Attack</u> ¹
1H	A	.962
1L	I	.9
2H	A	1
2L	I	1
3H	D	1
3L	S	1

The optimal policy is then as follows: if the contact is held at range 1 or 2, then attack; if the contact is lost at range 1 or 2, then increase the range by one unit; if the contact is held at range 3, then decrease the range by one unit; and if the contact is lost at range 3, then stay at range 3.

This example demonstrates the utility of the models presented in this chapter. They may be used to describe tactical situations where the searcher has some information on the position or status of the submarine. Of course, the inclusion of the states in the model permits the use of such tactical information.

We have used the perfect information model of the last section to describe a situation with two terminal states. This extension was

¹The numbers in this example are somewhat optimistic.

possible because we were maximizing the probability of termination in state S_0 . If, instead, we wished to minimize the time to arrive at state S_0 , then the model will require some non-trivial modifications. In the next section, we modify the perfect information model to include more than one terminal state in general.

4.10 Two Terminal States

We consider a game with perfect information and two terminal states. One of these terminal states, state 1, is favorable for P1 and the other terminal state, state n , is unfavorable for P1. When the pursuit has not terminated, it is in exactly one of the states $i = 2, \dots, n-1$. P1's objective is to minimize the expected cost of arriving at state 1. For example, the pursuit may terminate when P1 catches P2 or when P2 gets away from P1. Here P1 is interested in minimizing the cost of catching P2. Notice that we cannot achieve this objective with the model of section 4.8 because, in general, this model would simply minimize the cost of termination in either state 1 or n .

We develop a model with the already indicated objective. A device first introduced by Derman [1] is used to transform the given absorbing Markov chain to an equivalent irreducible chain. The cost of absorption in state 1 can then be expressed in terms of steady state probabilities. The problem of finding an optimal policy becomes a linear fractional programming problem, and this program is reduced to a linear program by the method of Charnes and Cooper [6]. When state n is deleted, the resulting linear program is precisely the dual of the program given in section 4.8. Hence, the approach taken here is, in a sense, "dual" to the approach used in section 4.8.

As before, we define the following parameters for states

$i = (2, \dots, n-1)$. Let a_i^k be the cost of decision k when the pursuit is in state i and let p_{ij}^k be the probability of transition to state j ($j = 1, \dots, n$), given state i and decision k ($k = 1, \dots, M_i$). The p_{ij}^k must satisfy

$$p_{ij}^k \geq 0 \quad \sum_{j=1}^n p_{ij}^k = 1 \quad \begin{matrix} i = 2, \dots, n-1 \\ k = 1, \dots, M_i \end{matrix}$$

We will define the transition probabilities and costs for states 1 and n later.

We introduce the set S of all randomized decision policies

$D = \{D_{ik}\}$ where D_{ik} is the probability of decision k given state i . We require

$$(42) \quad \sum_{k=1}^{M_i} D_{ik} = 1$$

Then, the stationary transition probabilities, p_{ij}^1 , and the cost of passing through state i , a_i , for each fixed $D \in S$, is

$$(43) \quad p_{ij} = \sum_{k=1}^{M_i} D_{ik} p_{ij}^k \quad i, j = 2, \dots, n-1; \quad a_i = \sum_{k=1}^{M_i} D_{ik} a_i^k \quad i = 2, \dots, n-1$$

We require the following assumption:

Assumption A: From each state i ($i = 2, \dots, n-1$) and for all

p_{ij}^1 is the probability of transition to state j given state i for some fixed $D \in S$.

D&S, state 1 is reached in a finite number of steps with probability one.

We expect Assumption A to hold in practical situations. For, policies are not permitted which result in (i) cycling between states or (ii) termination in state n only. Policies of type (i) or (ii) yield an infinite cost of arriving at state 1. Hence, we do not restrict the model by ruling out these possibilities.

We introduce a device due to Derman [1] to transform the absorbing Markov chain into an equivalent irreducible chain. Accordingly, for all D&S, we define the following transition probabilities of states 1 and n .

$$(44) \quad \begin{aligned} p_{1i} &= \frac{1}{n-2}, & p_{ni} &= \frac{1}{n-2} & i &= 2, \dots, n-1 \\ p_{1i} &= 0, & p_{ni} &= 0 & i &= 1, n \end{aligned}$$

The transition probability matrix for each fixed D&S is then

	1	2				n-1	n
1	0	$\frac{1}{n-2}$	$\frac{2}{n-2}$	\dots		$\frac{1}{n-2}$	$\frac{1}{n-2}$
2							0
n-1					P_{ij}		
n	0	$\frac{1}{n-2}$	$\frac{1}{n-2}$			$\frac{1}{n-2}$	$\frac{1}{n-2}$

As may be noted, when the pursuit arrives at either state 1 or n , it is started over again in one of the states $i = 2, \dots, n-1$ with uniform probability.

From Assumption A and equation (44), it is clear that the set

of states $\{1, \dots, n\}$ is irreducible for all D&S. We will depend heavily on this fact to formulate the objective function and the model. Let ${}^r p_{ij}$ be the r -step transition probability from state i to j ($i, j = 1, \dots, n$).

Consider a pursuit which lasts exactly $m \geq 1$ steps and for fixed D&S evolves according to the transition probabilities given by equations (43) and (44). Let $M_D(m, i)$ be the expected number of occurrences of state 1 when such a pursuit starts in state i , and let $C_D(m, i)$ be the expected cost of termination in state 1. When the pursuit starts in state i , the expected cost of the r th step is

$$\sum_{j=1}^n {}^r p_{ij} a_j \quad i = 1, \dots, n.$$

Hence, $C_D(m, i)$ is given by

$$C_D(m, i) = \frac{\sum_{r=1}^m \sum_{j=1}^n {}^r p_{ij} a_j}{M_D(m, i)}$$

and m is sufficiently large to insure that $M_D(m, i) \neq 0$. Let

$$C_D = \lim_{m \rightarrow \infty} C_D(m, i) \quad i = 1, \dots, n \text{ all D\&S.}$$

Theorem 6 establishes that the above limit exists and is independent of the starting state i . Notice that C_D is the expected cost of termination in state 1 regardless of the starting state i . We seek to find a D&S which minimizes C_D over all D&S.

Theorem 6: The expected cost of termination in state 1, starting

from state i and for each fixed policy $D \in S$, is independent of i ($i = 1, \dots, n$) and is given by

$$(45) \quad C_D = \frac{1}{\Pi_1} \sum_{j=1}^n \Pi_j a_j$$

where the Π_i are the unique solution to

$$(46) \quad \begin{aligned} \Pi_j - \sum_{i=1}^n \Pi_i p_{ij} &= 0 \quad j = 1, \dots, n \\ \sum_{j=1}^n \Pi_j &= 1 \end{aligned}$$

Proof

$$\begin{aligned} C_D &= \lim_{m \rightarrow \infty} C_D^{(m, i)} = \lim_{m \rightarrow \infty} \left[\frac{m}{M_D(m, i)} \frac{\sum_{r=1}^m \sum_{j=1}^n r p_{ij} a_j}{m} \right] \\ &= \left[\lim_{m \rightarrow \infty} \frac{m}{M_D(m, i)} \right] \left[\lim_{m \rightarrow \infty} \frac{\sum_{r=1}^m \sum_{j=1}^n r p_{ij} a_j}{m} \right] \end{aligned}$$

provided both of the limits in the above product exist.

But,

$$\lim_{m \rightarrow \infty} \frac{\sum_{r=1}^m \sum_{j=1}^n r p_{ij} a_j}{m} = \sum_{j=1}^n a_j \lim_{m \rightarrow \infty} \left[\frac{\sum_{r=1}^m r p_{ij}}{m} \right]$$

Since the set of states $(1, \dots, n)$ is irreducible for each $D \in S$, the Mean Ergodic theorem holds, i. e.,

$$\lim_{m \rightarrow \infty} \frac{\sum_{r=1}^m r P_{ij}}{m} = \Pi_j \quad j = 1, \dots, n$$

This limit is independent of i ($i = 1, \dots, n$) and the Π_j are the unique solution to (46). (These Π_j 's also satisfy $\Pi_j > 0$, $j = 1, \dots, n$)

Now

$$\lim_{m \rightarrow \infty} \frac{\sum_{r=1}^m \sum_{j=1}^n r P_{ij} a_j}{m} = \sum_{j=1}^n \Pi_j a_j$$

We also have¹

$$\lim_{m \rightarrow \infty} \frac{m}{M_D(m, i)} = \frac{1}{\Pi_1}$$

or $\frac{1}{\Pi_1}$ is the mean recurrence time of state 1. This limit is well

defined, since Assumption A guarantees $\Pi_1 > 0$.

Putting the above results together, we get

$$C_D = \frac{1}{\Pi_1} \sum_{j=1}^n \Pi_j a_j$$

This completes the proof.

We want to find a $D^* \in S$ which minimizes C_D over all $D \in S$.

From Theorem 6 and equations (42) and (43), D^* is an optimal solution

¹See Parzen [1].

to the following nonlinear programming problem.¹

$$\begin{aligned}
 \text{Min } & \frac{1}{\Pi_1} \sum_{j=1}^n \sum_{k=1}^{M_i} \Pi_j D_{jk} a_j^k \\
 & \Pi_j - \sum_{i=1}^n \sum_{k=1}^{M_i} \Pi_i D_{ik} p_{ij}^k = 0 \quad j = 1, \dots, n \\
 (47) \quad & \sum_{j=1}^n \Pi_j = 1 \\
 & \sum_{k=1}^{M_i} D_{jk} = 1 \quad j = 1, \dots, n \\
 & \Pi_j \geq 0, \quad D_{jk} \geq 0
 \end{aligned}$$

We transform problem (47) into a linear fractional programming problem by means of the following change of variables. Let

$$(48) \quad x_{jk} = \Pi_j D_{jk} \quad j = 1, \dots, n; \quad k = 1, \dots, M_j$$

From (48) and $\sum_{k=1}^{M_j} D_{jk} = 1$, we get

$$\Pi_j = \sum_{k=1}^{M_j} x_{jk} \quad j = 1, \dots, n.$$

Problem (47) becomes

¹For convenience, in the following formulation, we have $k = 1$ for states 1 and n and p_{ij}^1 ($i = 1, n$) is then given by equation (44).

$$(49) \quad \text{Min} \quad \frac{\sum_{j=1}^n \sum_{k=1}^{M_j} x_{jk} a_j^k}{\sum_{k=1}^{M_1} x_{1k}}$$

$$\text{Subject to:} \quad \sum_{k=1}^{M_j} x_{jk} - \sum_{i=1}^n \sum_{k=1}^{M_i} x_{ik} p_{ij}^k = 0 \quad j = 1, \dots, n$$

$$(49a) \quad \sum_{j=1}^n \sum_{k=1}^{M_j} x_{jk} = 1$$

$$x_{jk} \geq 0$$

Clearly the transformation (48) is one-to-one between optimal solutions to (47) and (49). Hence, we may solve (47) by solving (49).

We use the method of Charnes and Cooper [6] to transform problem (49) to an equivalent linear program. To establish this equivalence, we observe that the convex set of feasible solutions to (49) is

bounded and non-empty. Further, $\prod_1 = \sum_{k=1}^{M_1} x_{1k} > 0$ for all feasible

solutions to (49). Hence, the following transformation is one-to-one between problems (49) and (51).

$$(50) \quad y_{jk} = tx_{jk}, \quad \sum_{j=1}^n \sum_{k=1}^{M_j} y_{jk} = t, \quad t \sum_k x_{1k} = 1$$

Using (50), the equivalent linear program to (49) is¹

$$\begin{aligned}
 & \text{Min } \sum_{j=1}^n \sum_{k=1}^{M_j} y_{jk} a_j^k \\
 (51) \quad & \sum_{k=1}^{M_j} y_{jk} - \sum_{i=1}^n \sum_{k=1}^{M_i} y_{ik} p_{ij}^k = 0 \quad j = 1, \dots, n \\
 & \sum_{k=1}^{M_1} y_{1k} = 1 \\
 & y_{jk} \geq 0
 \end{aligned}$$

We make one further reduction of problem (51). Actually, there are no decisions to be made when the pursuit is in the terminal states $i = 1$ or n . Hence, we eliminate the variables y_{1k} and y_{nk} from problem (51). By means of equation (44) and some algebra, problem (51) is equivalent² to

$$\begin{aligned}
 & \text{Min } \sum_{j=2}^{n-1} \sum_{k=1}^{M_j} y_{jk} a_j^k \\
 (52) \quad & \sum_{k=1}^{M_j} y_{jk} - \sum_{i=2}^{n-1} \sum_{k=1}^{M_i} y_{ik} (p_{ij}^k + p_{in}^k) = \frac{1}{n-2} \quad j = 2, \dots, n-1 \\
 & \sum_{i=2}^{n-1} \sum_{k=1}^{M_i} y_{ik} p_{i1}^k = 1 \\
 & y_{ik} \geq 0
 \end{aligned}$$

¹The constraint (49a) becomes $\sum_{j=1}^n \sum_{k=1}^{M_j} y_{ik} = t$, since this constraint is redundant in (51) we have omitted it.

²For convenience, we have taken $a_1^k = a_n^k = 0$.

From problem (52), we obtain a final result concerning the nature of the optimal decision policy. The optimal policy is characterized by

$$(53) \quad D_{jk}^* = \begin{cases} 1 & \text{for } k = k_j \\ 0 & \text{for } k \neq k_j \end{cases} \quad j = 2, \dots, n-1$$

where k_j is some decision for state j .

Of course, (53) says that a "pure policy" is optimal, i. e., for each state pick some alternative with probability one. Equation (53) follows from the following observations. From the constraints of (52)

an optimal solution $\{y_{jk}^*\}$ satisfies $\sum_{i=2}^{n-1} \sum_{k=1}^{M_i} y_{ik}^* (p_{ij}^k + p_{in}^k) \geq 0$ and, hence,

$$(54) \quad \sum_{k=1}^{M_j} y_{jk}^* > 0 \quad j = 2, \dots, n-1$$

Now one of the equality constraints in (52) is redundant. This may be verified by summing over the first $n-2$ constraints. Hence, (52) has, at most, $n-2$ linearly independent constraints (excepting non-negativity conditions); and, hence, a basic feasible solution has at most $n-2$ positive variables. Further, at least one basic feasible solution must be optimal. By (54) and the fact that at most $n-2$ variables can be positive in an optimal solution, we have

$$\begin{aligned}
 y_{jk}^* &> 0 \quad \text{for } k = k_j \\
 y_{jk}^* &= 0 \quad \text{for } k \neq k_j
 \end{aligned}
 \quad j = 2, \dots, n-1$$

where k_j is some decision for state j .

Now, by the transformations set up between the D_{ik}^* variables and the y_{jk}^* variables, we conclude that our assertion (53) is correct.

We have shown how to formulate a model for situations involving two terminal states and the objective of minimizing the cost to arrive at one of these terminal states. In section 4.9, we gave an example of a two terminal state situation. If our objective for that problem was to minimize the time to complete a successful attack, then the model presented in this section is applicable. All immediate payoffs, a_i^k , are taken equal to one to achieve the "time" objective. As may be noted, the approach taken here results in a linear program which is the dual of the linear program obtained from the functional equation approach.

4.11 A Finite Terminating Stochastic Game

We return to our idealization of ASW and introduce the additional rule: the pursuit is terminated in a specified finite number of moves if it has not already reached a terminal state. In ASW, this forced termination may be caused by one of a number of factors, such as resource limitations or submarine endurance time when submerged. This means we have a finite version of a TSG. It terminates in m moves or a terminal state, whichever occurs first.

The following notation is introduced for the finite TSG. Consider a collection of mutually exclusive and collectively exhaustive states numbered $i = 1, \dots, N$. Terminal states are included in this collection, and the finite TSG must be in one and only one of these states at each move $t = 1, \dots, m$. When the game is in state i , we number the available alternatives for players 1 and 2 respectively, $k = 1, \dots, M_i$ and $h = 1, \dots, N_i$. If the finite TSG is in state i at move t and players 1 and 2 choose alternatives k and h respectively, then the payoff from player 2 to player 1 is

$$a_i^{kh}(t), \quad \begin{array}{ll} i = 1, \dots, N & k = 1, \dots, M_i \\ t = 1, \dots, m & h = 1, \dots, N_i \end{array}$$

and the game transits to state j with probability

$$p_{ij}^{kh}(t), \quad \begin{array}{ll} i, j = 1, \dots, N & k = 1, \dots, M_i \\ t = 1, \dots, m & h = 1, \dots, N_i \end{array}$$

Since the states are assumed to be mutually exclusive and collectively exhaustive, the $p_{ij}^{kh}(t)$ must satisfy

$$\sum_{j=1}^n p_{ij}^{kh}(t) = 1 ; \quad \text{also, } p_{ij}^{kh}(t) \geq 0, \quad \text{all } i, j, k, h, t.$$

Finally, we assume that the players are informed of both the state and the move before they choose their strategies.

One will note that the above structure is different from that of the infinite TSG in the following respects. In the finite TSG:

1. The payoffs and transition probabilities may depend on the move.
2. There may be a zero probability of termination in one move.
3. The players know the state of the game and the move when they choose their strategy for the next move.

Thus, if the game is finite, more flexibility may be permitted in the model, i. e., items 1 and 2 above.

Next, we show how the value and optimal strategies of a finite TSG may be computed. As will be noted, the methods and representations developed here are closely related to those of the infinite game.

Let $v_i(t)$, $\left\{ \begin{matrix} i = 1, \dots, N \\ t = 1, \dots, m \end{matrix} \right\}$ be the minimax of the total expected payments received by player 1 from the remaining $m - t$ moves when the game is in state i at move t , and let $V(t) = (v_1(t), \dots, v_N(t))$. Now, $V(m)$ is the minimax of the total expected payments with zero moves to go; accordingly, $V(m) = 0$. For convenience, we introduce the $M_i \times N_i$

matrix $A_{it}(a)$ whose k - h^{th} element is

$$a_i^{kh}(t) + \sum_{j=1}^N p_{ij}^{kh}(t) a_j \quad \begin{matrix} k = 1, \dots, M_i \\ h = 1, \dots, N_i \end{matrix}$$

The minimax of the expected payments with one move left, $V(m-1)$, is clearly given by the following set of equations:

$$V_i(m-1) = \text{Val}[A_{im}(V(m))] \equiv \text{Val}[A_{im}(0)], \quad i = 1, \dots, N.$$

Let $\hat{X}_i(m)$ and $\hat{Y}_i(m)$ be optimal strategies for players 1 and 2 respectively in the game $A_{im}(V(m))$. Then it follows that

$\hat{X}(m) = (\hat{X}_1(m), \dots, \hat{X}_N(m))$ and $\hat{Y}(m)$ are optimal strategies in the m^{th} move of the finite TSG. Since the payoffs and transition probabilities depend only on the move and the state, which are known to the players, it may be established by induction that the following relationship obtains:

$$(54) \quad \begin{matrix} i = 1, \dots, N \\ t = 1, \dots, m \end{matrix} \quad V_i(t-1) = \text{Val}[A_{it}(V(t))].$$

According to equation (54), $V(0)$ is the value of the finite TSG.

Let $\hat{X}_i(t)$ and $\hat{Y}_i(t)$ be optimal strategies for players 1 and 2 re-

spectively in the game $A_{it}(V(t)) \left\{ \begin{matrix} i = 1, \dots, N \\ t = 1, \dots, m \end{matrix} \right\}$. Then it is clear, from equation (54), that $\hat{X}(t) = (\hat{X}_1(t), \dots, \hat{X}_N(t))$ and

$\hat{Y}(t) = (\hat{Y}_1(t), \dots, \hat{Y}_N(t))$ are optimal strategies in move t of the finite TSG. Note that $\hat{X}(t)$ and $\hat{Y}(t)$ depend, in general, on the move of the game and are, therefore, not behavior strategies. In general,

behavior strategies are not optimal in a finite TSG.

Returning to the linear program L. P. $(i, V(t))$, we see that it is a linear programming formulation of the game $A_{it}(V(t))$ with the payoffs and transition probabilities depending on t . To compute the value and optimal strategies for the finite TSG, we can start with $V(m) = 0$ and compute $\hat{X}(m)$, $\hat{Y}(m)$, and $\hat{U}(m)$ from $\{L. P. (i, V(m))\}$, $i = 1, \dots, N$. Now, set $V(m-1) = \hat{U}(m)$ and compute $\hat{X}(m-1)$, $\hat{Y}(m-1)$, and $\hat{U}(m-1)$, and so on. Thus, the value and optimal strategies may be computed recursively by linear programming.

CHAPTER V - MULTIPLE CONTACT ALLOCATION MODELS

5.1 Introduction

We develop models for the allocation of hunter-killer forces to multiple contact areas. The central problem is to determine an optimal division of effort, between several contacts, subject to typical constraints. To focus attention on the ideas, we consider hunter-killer operations which consist of at least two separate contact areas. For the first model types, we assume that effort is allocated to the contact areas only once during the planning horizon. We then relax this restriction and formulate a dynamic allocation model. Now, before introducing these models, we discuss the predominate features of the tactical situations which will be considered.

In a wide variety of military problems, the force level required to accomplish a given military mission is uncertain when the allocation of forces is made. One of the primary causes of this uncertainty is due to lack of information on enemy forces and capabilities. To reflect this uncertainty in the model we assume that the amount of effort which is required to accomplish a specified military mission in each area is a random variable with a known joint cumulative distribution function (c. d. f.). This c. d. f. may be rather difficult to determine in

practice! Nevertheless, we assume that it can be determined and sensitivity studies can then be conducted to determine the effects of estimation errors and data variations. After the models are formulated, we can also ascertain the effect of treating the random requirements as deterministic quantities.

Next, we introduce the objective functions which will be employed. These objectives are oriented toward optimizing a measure of overall mission success. From the specification of the random requirements, we can readily relate individual mission success to overall effectiveness. For we have assumed that the military mission in each tactical area can be accomplished if the allocated force level exceeds the observed random level. Hence, our objective functions measure the "difference" between allocated levels and the random requirements. In particular, the following two objectives are used:

1. Maximize the probability that all allocated force levels simultaneously exceed their random requirements. This is equivalent to maximizing the probability that all missions are simultaneously accomplished.
2. Minimize the total expected shortage between allocated and required levels.

In some situations, time may be an important measure of effectiveness. For these cases, the following objective is employed:

3. Minimize the expected distribution time to achieve a specified probability that all requirements are met.

Each of the above objectives will be studied for the "one-shot" allocation models. Objective (3) is the only one which is employed for the dynamic model.

We have introduced the requirements and objectives which will be taken for multiple contact situations. Now we discuss a measure of available effort. We measure the available effort in some meaningful unit such as a ship, surface attack unit, or one flying hour. If several types of effort are available, then all effort is measured in terms of a single "standard" unit. However, the models could be extended to include different types of effort. Depending on the measurement adopted, effort may be treated as continuous or discrete. For instance, if a unit of effort is one flying hour, then effort may be treated as continuous. On the other hand, a unit of one ship will usually require a discrete treatment. Both discrete and continuous measurements will be studied for most of our models.

5.2 The Probability Model

We formulate a model with objective (1) of the last section. Methods of solving the model are given and the detailed solution for a uniform distribution is presented. This is to be a "one-shot" allocation problem; only one allocation is made to each contact area.

We assume that n contact areas are specified and d_j is the amount of effort required in area j ($j = 1, \dots, n$) to accomplish the mission there. Further, each d_j is a random variable and the joint cumulative distribution function (c. d. f.) of d_1, d_2, \dots, d_n is assumed known. We let F be this joint c. d. f., then

$$F(y_1, y_2, \dots, y_n) = \Pr(d_1 \leq y_1, d_2 \leq y_2, \dots, d_n \leq y_n),$$

where "Pr" denotes probability. Let x_j be the amount of effort to be assigned to area j ($j = 1, \dots, n$) and let "a" be the total amount of effort available. Now objective (1) of section 5.1 gives way to the following optimization problem:

$$\begin{aligned} (1) \quad & \text{Max } F(x_1, x_2, \dots, x_n) \\ & \text{Subject to: (1a) } \sum_{j=1}^n x_j \leq a \\ & \quad \quad \quad x_j \geq 0 \end{aligned}$$

The objective is to maximize the joint probability that all allocations x_j exceed their random requirements d_j . Of course, the fundamental notion that d_j is the amount of effort required to accomplish the

mission in area j leads to the interpretation that we are maximizing the probability that all missions are simultaneously accomplished. The restrictions in model (1) are on the total amount of effort available and on the non-negativity of each individual allocation x_j .

The burden of optimization in (1) is placed on the objective function. Later, we consider models with more complicated constraints and a simpler objective function. This model embodies the essentials of H. A. Simon's [1] satisficing approach. For we maximize the probability that a specified goal is reached. In addition, this objective is similar to the one used by Charnes and Cooper [4] in their so-called "P-model". Next, we discuss methods for solving problem (1).

We apply the Kuhn - Tucker conditions of convex programming to (1). To employ these conditions, we require that F is continuously differentiable. Without loss of generality, we may replace the inequality in (1.1) by equality, since F is a c. d. f. and therefore it is a monotone non-decreasing function. The Kuhn - Tucker necessary conditions are the following:¹ if $X = (x_1, x_2, \dots, x_n)$ is an optimal solution to (1), then there exists a scalar μ such that X and μ satisfy

¹Of course, "constraint qualification" is satisfied by the constraints of (1).

$$(2) \quad \frac{\partial F}{\partial x_j} - \mu \leq 0 \quad j = 1, \dots, n$$

$$(3) \quad x_j \frac{\partial F}{\partial x_j} - \mu = 0 \quad j = 1, \dots, n$$

$$(4) \quad \sum_{j=1}^n x_j = a, \quad x_j \geq 0 \quad j = 1, \dots, n$$

If F is a concave function, then the above conditions are also sufficient for X to be an optimal solution to (1). These equations are rather difficult to solve in general because (2) and (3) are usually non-linear. Nevertheless, we will apply these conditions to a special case of (1), but first we examine model (1) when d_1, \dots, d_n are independently distributed.

Numerous tactical problems have independent d_1, d_2, \dots, d_n . We would expect independence when an allocation to one area does not have an appreciable spillover effect on other areas. Indeed, contact areas are often widely separated and no interaction occurs between areas. Furthermore, independence would probably be required in order to empirically determine F . With the assumption that d_1, \dots, d_n are independently distributed, we obtain

$$(5) \quad F(x_1, \dots, x_n) = F_1(x_1) F_2(x_2) \dots F_n(x_n)$$

$$\text{where } F_j(x_j) = \Pr(d_j \leq x_j)$$

Now, we may maximize the logarithm of the function in (5),

since the log is a monotone transformation and (5) is non-negative. With this transformation, problem (1) becomes

$$\begin{aligned} \text{Max } & \sum_{j=1}^n \log F_j(x_j) \\ (6) \quad \text{Subject to: } & \sum_{j=1}^n x_j = a \\ & x_j \geq 0 \end{aligned}$$

Perhaps the most general method which is available to solve (6) is dynamic programming. It is especially useful when the x_j are required to be non-negative integers. Since the application of dynamic programming to allocation problems has been extensively studied (see Bellman and Dreyfus [1]), we do not dwell on this method here. Instead, we turn to an important special case.

The special case is studied where the random variables d_j are independent and uniformly distributed between a_j and b_j . A uniform distribution implies that the actual requirement occurs at random. Roughly speaking, no particular requirement is preferred over any other requirement.

With the uniform distribution and independence assumption, we have

$$(7) \quad F_j(x_j) = \begin{cases} 0 & x_j < a_j \\ \frac{x_j - a_j}{b_j - a_j} & a_j \leq x_j < b_j \\ 1 & x_j \geq b_j \end{cases}$$

Notice that when $\sum_{j=1}^n a_j \geq a$ then the maximum in (6) is negative infinity

and when $\sum_{j=1}^n b_j \leq a$ then all solutions with each $x_j \geq b_j$ are optimal

Hence, we restrict our attention to the following non-trivial case:

$$(8) \quad \sum_{j=1}^n a_j < a < \sum_{j=1}^n b_j$$

For convenience, we introduce the transformation

$$(9) \quad y_j = x_j - a_j$$

and we let $c = a - \sum_{j=1}^n a_j > 0$, $c_j = b_j - a_j > 0$

then, when (8) is satisfied, problem (6) with equation (7) is equivalent to

$$(10) \quad \begin{aligned} \text{Max } & \sum_{j=1}^n \log y_j \\ & \sum_{j=1}^n y_j = c \\ & 0 \leq y_j \leq c_j \end{aligned}$$

Now the optimal solution to (10) may be readily obtained by the following method. For simplicity, suppose $c_1 \leq c_2 \leq \dots \leq c_n$. Then the following allocation is optimal

$$y_1 = \min(c_1, \frac{c}{n})$$

$$(11) \quad y_i = \begin{cases} \min(c_i, \frac{c - \sum_{j=1}^{i-1} c_j}{n - i + 1}) & \text{if } y_{i-1} = c_{i-1} \\ y_{i-1} & \text{if } y_{i-1} \neq c_{i-1} \end{cases} \quad i = 2, \dots, n$$

The optimality of (11) can be routinely verified by showing that this solution satisfies the Kuhn - Tucker sufficient conditions. The procedure given by (11) allocates an equal amount to each activity until $y_1 = c_1$. Then equal amounts are allocated to the remaining activities until $y_2 = c_2$ or $\sum_{i=1}^n y_i = c$. This process is continued until $\sum_{i=1}^n y_i = c$.

To illustrate, consider the following simple example:

$$c_1 = 2, c_2 = 3, c_3 = 5, c_4 = 9, c = 12$$

The optimal solution from (11) is

$$y_1 = 2, y_2 = 3, y_3 = y_4 = 7/2$$

We examine the tactical consequences of the solution given by (11). This optimal solution requires a maximal allocation of c_j to certain areas, namely those areas where a probability of one can be achieved with the least amount of effort. All other areas which have

not achieved a maximal allocation receive the same amount of effort which is greater than the largest c_j for those areas which have achieved a probability of one.

This policy is appealing in some aspects but it has drawbacks introduced by the linearity of the c. d. f. 's, namely maximal allocation to some areas. This phenomena would disappear with c. d. f. 's of the non-linear type such as those of the exponential family. Rather than pursue these points further here, we turn to the second objective of minimizing the expected shortages.

5.3 Expected Shortage Model

We formulate the expected shortage model and show that the case of independent uniform distributions is reduced to a quadratic programming problem. The tactical motivation for this model is the same as for the one of the last section. Hence, we proceed directly with the formulation of the model.

Our objective will be to minimize the expected excess of demand over supply (allocation). Therefore, we introduce the following shortage function.

$$(12) \quad \phi_j(x_j, Z) = \begin{cases} 0 & x_j > Z \\ Z - x_j & x_j \leq Z \end{cases} \quad j = 1, \dots, n$$

where as before x_j is the amount allocated to area j . Then $\phi_j(x_j, Z)$ is the shortage in area j , if Z is the actual demand. Let F_j be the marginal c. d. f. of d_j and let $E_j(x_j)$ denote the expected value of $\phi_j(x_j, Z)$. Now we assume that each F_j is sufficiently well-behaved so that Stilges integration by parts may be performed. Then,

$$\begin{aligned} E_j(x_j) &= \int_{-\infty}^{\infty} \phi_j(x_j, Z) dF_j(Z) = \int_{x_j}^{\infty} (Z - x_j) dF_j(Z) \\ &= \int_0^{\infty} (Z - x_j) dF_j(Z) - \int_0^{x_j} (Z - x_j) dF_j(Z) \end{aligned}$$

$$\begin{aligned}
 E_j(x_j) &= \mu_j - x_j - \left[(Z - x_j) F_j(Z) \Big|_{Z=0}^{x_j} - \int_0^{x_j} F_j(Z) dZ \right] \\
 &= \mu_j - x_j - x_j F_j(0) + \int_0^{x_j} F_j(Z) dZ
 \end{aligned}$$

where μ_j is the mean of d_j .

We have $F_j(0) = 0$, since d_j is a non-negative random variable.

Hence,

$$(13) \quad E_j(x_j) = \mu_j - x_j + \int_0^{x_j} F_j(Z) dZ$$

According to our stated objective 2 from section 5.1 and the previously indicated constraints, we formulate the following optimization problem:

$$\begin{aligned}
 (14) \quad (14a) \quad & \text{Min } \sum_{j=1}^n \lambda_j E_j(x_j) \\
 & \sum_{j=1}^n x_j \leq a \\
 & x_j \geq 0
 \end{aligned}$$

where the λ_j are specified weighting factors with $\lambda_j \geq 0$, $\sum_{j=1}^n \lambda_j = 1$.

These weighting factors may be used to reflect the relative importance of shortages between various areas.

Problems similar to (14) have been investigated by other authors. For example, Charnes, Cooper, and Thompson [2] investigate a general class of problems in "constrained generalized medians"¹ and (14)

¹ See this reference for an extensive list of references.

belongs to this class.

Now $E_j(x_j)$ is a convex-decreasing function of x_j and, therefore, the inequality (14a) may be replaced by an equality. There are two general solution techniques which may be applied to (14). The first technique is dynamic programming. Of course, this method is especially useful when the x_j are required to be non-negative integers. Dynamic programming does not require many special properties of the function F_j and it is an efficient computational technique. (See Bellman and Dreyfus [1].) The other technique which may be used to solve (14) is the Charnes and Lemke [1] minimization technique for non-linear separable convex functions. This technique is especially useful when more constraints are adjoined to (14).

For purposes of comparison with the previous section, we discuss the special case when the d_j are independent and uniformly distributed. Accordingly, we assume that $F_j(x_j)$ is given by equation (7). Then $E_j(x_j)$ will take the following form

$$E_j(x_j) = \begin{cases} \mu_j - x_j, & 0 \leq x_j < a_j \\ \mu_j - x_j + \frac{(x_j - a_j)^2}{2(b_j - a_j)}, & a_j \leq x_j < b_j \\ 0, & b_j \leq x_j \end{cases}$$

where $\mu_j = a_j + \frac{1}{2}(b_j - a_j)$.

As may be noted, $E_j(x_j)$ is either a linear or quadratic function of

x_j (see Figure 5.1 below).

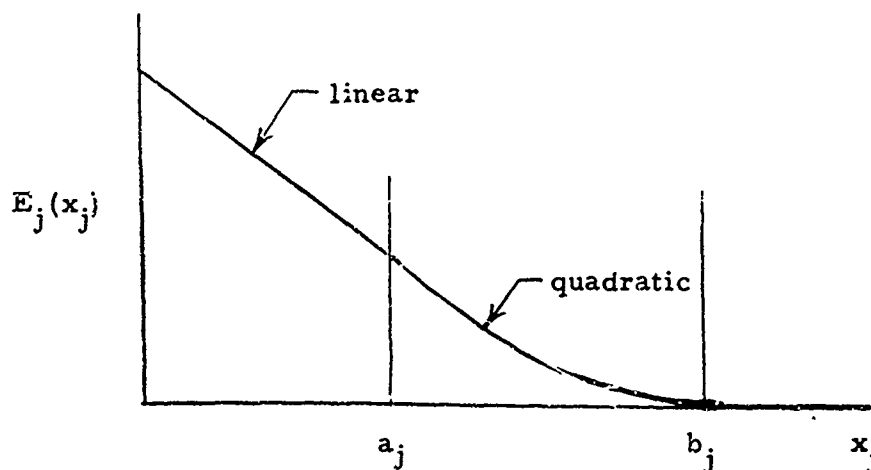


Figure 5.1

To reduce our model to a quadratic programming problem, we introduce the variables v_j and w_j with

$$\begin{aligned} x_j &= v_j + w_j \quad j = 1, \dots, n \\ (16) \quad 0 &\leq v_j \leq a_j \\ 0 &\leq w_j \leq c_j \quad \text{where } c_j = b_j - a_j \geq 0 \quad j = 1, \dots, n. \end{aligned}$$

However, we must require $w_j = 0$ when $v_j < a_j$. This is accomplished by the non-linear conditions

$$w_j (v_j - a_j) = 0 \quad j = 1, \dots, n.$$

These restrictions can be maintained by restricted basis entry. With the above change of variables, the expectation of (15) becomes

$$(17) \quad E_j = \mu_j - (v_j + w_j) + \frac{1}{2c_j} w_j^2$$

Problem (14) now gives way to the following quadratic programming problem.¹

$$\begin{aligned}
 (18) \quad & \text{Max } \sum_{j=1}^n \lambda_j \left[v_j + w_j - \frac{1}{2c_j} w_j^2 \right] \\
 & \sum_{j=1}^n v_j + w_j = a \\
 & w_j (v_j - a_j) = 0 \\
 & 0 \leq v_j \leq a_j \\
 & 0 \leq w_j \leq c_j
 \end{aligned}$$

Any of the standard quadratic programming methods can be used to solve (18). Of course, bounded variable techniques can also be employed to substantially reduce the size of the constraint set and thereby improve computational efficiency.

For the uniform case, we compare the expected shortage model to the probability model. We take $\lambda_j = \frac{1}{n}$ $j = 1, \dots, n$ in (18), since the probability model maximizes the probability that all missions are simultaneously accomplished and therefore the missions are equally weighted. Notice that we will have a nontrivial optimal solution for (18) when $\sum_{j=1}^n a_j \geq a$. This was not true for the probability model. To obtain

¹The constant $\sum \lambda_j \mu_j$ has been dropped from the objective function and the optimization has been changed to maximization by multiplying the objective function by -1. We have also excluded the trivial case $\sum b_j \leq a$ and this permits us to write equality in the second constraint.

a direct comparison, we require "a" to satisfy

$$(19) \quad \sum_{j=1}^n a_j \leq a \leq \sum_{j=1}^n b_j$$

The optimal solution to (18) will then have all $v_j = a_j$, so we delete the v_j variables and the non-linear conditions $w_j(v_j - a_j) = 0$; then the w_j variables of (18) correspond directly to the y_j variables of (10). When (19) is satisfied, then (10) and (18) become

$$(10a) \quad \text{Max } \sum_{j=1}^n \log y_j$$

$$\sum_{j=1}^n y_j = c$$

$$0 \leq y_j \leq c_j$$

$$(18a) \quad \text{Max } \sum_{j=1}^n \left[w_j - \frac{1}{2c_j} w_j^2 \right]$$

$$\sum_{j=1}^n w_j = c$$

$$0 \leq w_j \leq c_j$$

In general, the optimal solutions to (10a) and (18a) will not be the same. However, they are the same if all c_j are equal. This in turn is true if and only if every mission has the same variance. In other cases it is difficult to obtain direct comparisons between (10a) and (18a) unless the constants c_j are known. But, since the constraint sets of (10a) and (18a) are identical, differences in optimal solutions are attributed to differences in the objective functions. Next, we take up a different approach to the multiple-contact problem via the constructs of chance-constrained programming.

5.4 Chance-Constrained Distribution Model

In this section we introduce a more complicated model of a multiple contact problem. This model also utilizes the concept of random demands which are required to accomplish a specified mission. However, here we explicitly recognize that the units of effort may come from different origins. The problem is to minimize the expected distribution time to accomplish each mission with at least a specified probability. We develop a chance-constrained distribution model of this problem.

To formulate the model, suppose that a_i units are available at some location (origin) i ($i = 1, \dots, m$), where a_i is a given non-negative integer. Further, assume that the number of units required to accomplish the specified mission at some location (destination) j ($j = 1, \dots, n$) is a non-negative discrete random variable d_j with the known marginal c. d. f., F_j . Of course, some of the origins and destinations may coincide. Also, F_j may be a degenerate distribution for some destinations, giving rise to a deterministic requirement.

The above assumptions lead directly to distribution type constraints. One set of constraints is as follows: the amount sent from any origin cannot exceed the amount available. Then, letting x_{ij} be the amount sent from origin i to destination j , we get

$$(20) \quad \sum_{j=1}^n x_{ij} \leq a_i \quad i = 1, \dots, n$$

The second set of constraints is written to reflect the random nature of the requirements. The value of the random variable d_j is observed after the allocations are made. In the face of this uncertainty, we employ the ingenious notion of chance-constraints due to Charnes and Cooper [2]. These constraints are¹

$$(21) \quad \Pr \left\{ \sum_{i=1}^m x_{ij} \geq d_j \right\} \geq c_j \quad j = 1, \dots, n$$

where the c_j are specified constants. The double inequality in (21) reads as follows: the number of units sent to destination j $\left(\sum_{i=1}^m x_{ij} \right)$ must exceed the actual requirement at least $100 c_j \%$ of the time. Hence, these constraints guarantee a stipulated level of protection against shortages at each destination. In addition, we will place non-negativity restrictions on the x_{ij} .

As mentioned previously, our objective will be to minimize the expected distribution time. Accordingly, we let t_{ij} be the time for one unit to travel from origin i to destination j , where the t_{ij} are random variables with known means. Then, our objective is

$$(22) \quad \text{Min } E \left\{ \sum_i \sum_j t_{ij} x_{ij} \right\}$$

¹"Pr" denotes probability in the following equation.

where "E" is the expectation operator. We let \hat{t}_{ij} be the expected value of t_{ij} . Then, bringing together the above objective and the already indicated constraints, we obtain the following optimization problem.¹

$$(23) \quad \begin{aligned} \text{Min } & \sum_i \sum_j \hat{t}_{ij} x_{ij} \\ & \sum_j x_{ij} \leq a_i \end{aligned}$$

$$(23a) \quad \begin{aligned} \text{Pr} \left\{ \sum_i x_{ij} \geq d_j \right\} & \geq c_j \\ x_{ij} & \geq 0 \end{aligned}$$

Problem (23) is distribution model with the chance constraints (23a). Fortunately, we can solve this model by obtaining an equivalent distribution model with no random elements, a deterministic equivalent.²

To proceed, we rewrite equation (23a) in terms of the known c. d. f. 's, i. e.

$$(24) \quad F_j(x_j) \geq c_j$$

where

$$x_j = \sum_i x_{ij}$$

Recall that d_j is a discrete random variable and let

$$p_{jk} = \text{Pr} \{d_j \leq k\} = F_j(k) \quad \begin{aligned} k &= 0, 1, 2, \dots \\ j &= 1, \dots, n \end{aligned}$$

¹To reduce the objective function, we use the fact that the expectation of a sum of random variables is the sum of the expectations.

²The approach used is due to Charnes and Cooper [4].

To obtain the deterministic equivalent, we define the function \bar{F}_j which is a pseudo inverse of F_j .

$$(25) \quad \bar{F}_j(y) = \begin{cases} 0 & y \leq 0 \\ k & p_{j, k-1} < y \leq p_{jk} \quad k = 0, 1, 2, \dots \end{cases}$$

For each j , the relationship between \bar{F}_j and F_j is depicted by Figure 5.2 below.

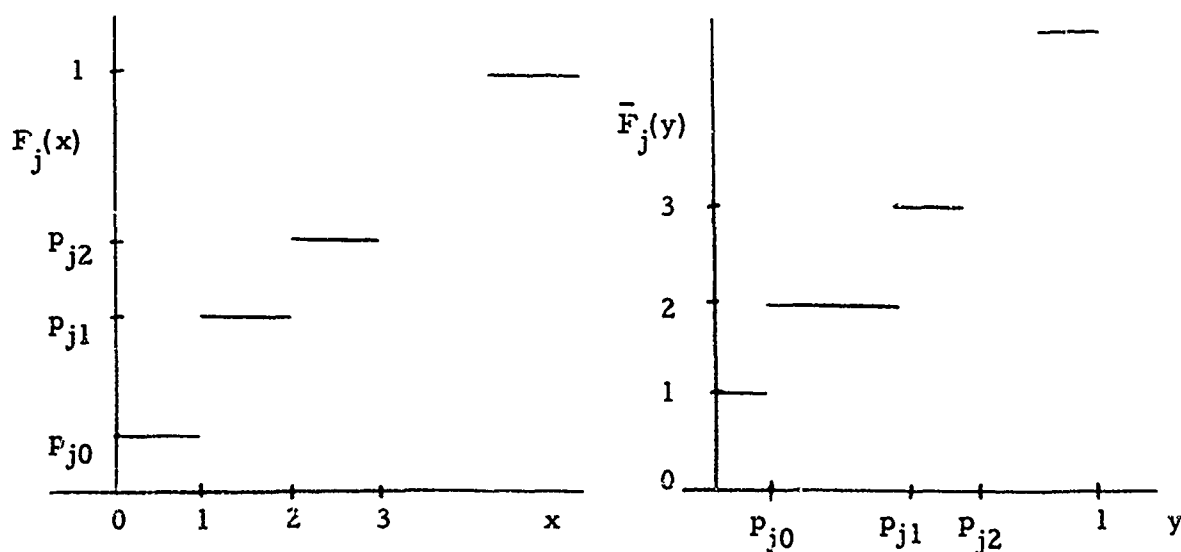


Figure 5.2

The following lemma provides the wanted reduction of equation (24).

Lemma 1: $F_j(x_j) \geq c_j$ if and only if $x_j \geq \bar{F}_j(c_j)$.

Proof: Assume that for some fixed j , x_j satisfies $F_j(x_j) \geq c_j$.

Then, since F_j is monotone non-decreasing, $x_j \geq k$ where k is the unique integer which satisfies

$$p_{j, k-1} < c_j \leq p_{jk}$$

Now, by definition of \bar{F}_j , $\bar{F}_j(c_j) = k$; hence,

$$x_j \geq \bar{F}_j(c_j)$$

To prove the opposite implication, assume for fixed j that x_j satisfies $x_j \geq \bar{F}_j(c_j)$. Then, $x_j \geq k$ where k is the unique integer which satisfies

$$p_{j, k-1} < c_j \leq p_{jk}$$

But, F_j is monotone non-decreasing; thus,

$$F_j(x_j) \geq F_j(k)$$

Furthermore,

$$F_j(k) = p_{jk} \geq c_j$$

Hence, $F_j(x_j) \geq c_j$ and the lemma is proved.

From Lemma 1, problem (23) and the following problem are equivalent

$$\begin{aligned} & \text{Min } \sum_i \sum_j t_{ij} x_{ij} \\ (26) \quad & \sum_j x_{ij} \leq a_i \\ (26a) \quad & \sum_i x_{ij} \geq \bar{F}_j(c_j) \\ & x_{ij} \geq 0 \end{aligned}$$

Since problem (26) is a distribution model, the following well-known properties¹ of (26) or equivalently (23) are immediately available:

¹See Charnes and Cooper [5].

1. Problem (26) has an optimal solution if and only if

$$\sum_i a_i \geq \sum_j \bar{F}_j(c_j)$$

2. Since the $\bar{F}_j(c_j)$ $j = 1, \dots, n$ are integers and the a_i $i = 1, \dots, m$ are assumed to be integers, (26) has an optimal integer extreme point solution.

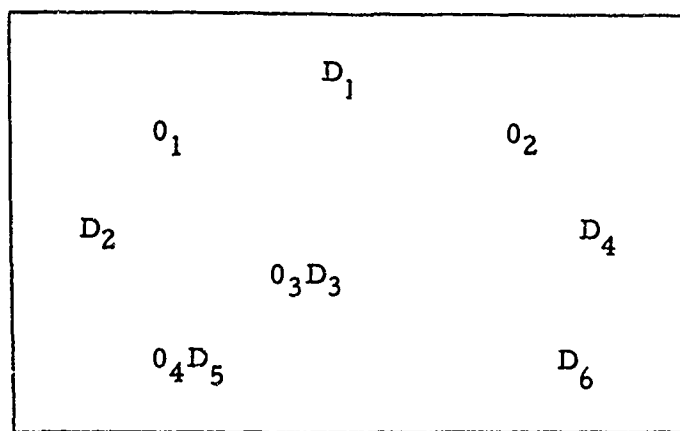
3. The inequalities in (26a) may be replaced by equalities without changing the values or the existence of optimal solutions (since all $\hat{t}_{ij} \geq 0$).

Property 2 above is especially useful because no special integer techniques are required to obtain an integer solution.

It is interesting to note that (26) is infeasible if there are not enough units available to obtain the stipulated confidence levels in (23). Of course, feasibility may be secured by reducing the value of some c_j or increasing the amounts available. One of the important features of (23) is that its deterministic equivalent (26) has a dual and, therefore, dual interpretations can be obtained. The dual evaluators indicate the change in the objective function per unit change in a_i or in $\bar{F}_j(c_j)$. This leads to an immediate evaluation of the effect of a change in c_j on the optimal solution. Of course, this effect is discontinuous since \bar{F}_j is a step function (see Figure 5.2). In addition to dual evaluation, sensitivity and parametric studies can also be implemented.

We conclude discussion of this model with an example. Suppose

that a known number of ships are available at each of four origins and that specified missions are to be accomplished at each of six destinations during the planning horizon. This initial tactical configuration is shown in Figure 5.3.



O_i denotes origin i and D_j denotes destination j .

Figure 5.3

Some of the above origins and destinations coincide since units are both available and required at these points. The tactical information pertaining to the destinations is given in the following Table 5.1. We also have computed $\bar{F}_j(c_j)$.

j	$F_j(k)$			c_j	$\bar{F}_j(c_j)$
	k=0	k=1	k=2		
D_1	0.2	0.6	1.0	.7	2
D_2	0.5	0.8	1.0	.9	2
D_3	0.4	0.9	1.0	.8	1
D_4	0	0	1.0	1.0	2
D_5	0.7	0.9	1.0	.6	0
D_6	0.7	0.9	1.0	.9	1

Demand Distributions

Table 5.1

The mean travel times from each origin to each destination and the amount available at each origin are given in Table 5.2 below.

\hat{t}_{ij}	D_1	D_2	D_3	D_4	D_5	D_6	Avail
0_1	2	3	4	6	8	10	3
0_2	2	4	4	3	7	8	2
0_3	4	3	0	3	3	4	2
0_4	8	5	3	5	0	3	2

Travel Times and Amounts Available

Table 5.2

As may be noted from Table 5.1, some destinations are more "critical" than others. For, certain requirements must be satisfied with higher probability. In this way, the constants, c_j , reflect the relative importance of the missions. Also from Table 5.1, a demand already exists at D_4 so that 2 units must be sent there with probability 1. Finally, no units are required at D_5 to attain the stipulated level of protection against shortages and D_5 is now deleted from the problem.

The data from Tables 5.1 and 5.2 is assembled in the following distribution tableau.

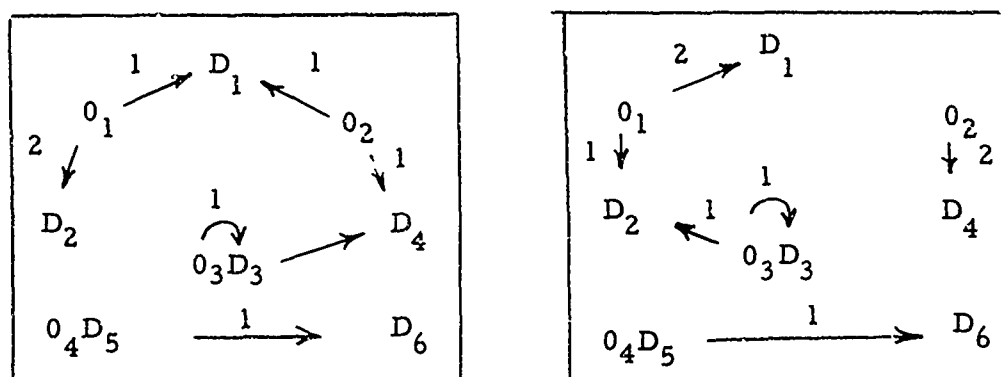
	D_1	D_2	D_3	D_4	D_6	Dummy	Avail
O_1	2 (1)	3 (2)	4	6	10	0	3
O_2	2 (1)	4	4	3 (1)	8	0	2
O_3	4	3	0 (1)	3 (1)	4	0 (6)	2
O_4	8	5	3	5	3 (1)	0 (1)	2
Req'd	2	2	1	2	1	1	

Optimal expected time = 19.

Optimal Tableau

Table 5.3

The optimal solution is circled in the preceding tableau. An alternate optimum is also available. Both optimal solutions are shown below.



Optimal Distribution Schedules

Figure 5.4

As we have mentioned, there are a number of sensitivity studies which can be conducted. Most of these are well understood, and we do not dwell on them here. However, we have computed the variation in c_j which is allowed before the requirements $\bar{F}_j(c_j)$ change. These computations are made from Table 5.1 and they are:

$$.6 < c_1 \leq 1.0, \quad .8 < c_2 \leq 1.0, \quad .4 < c_3 \leq .9$$

$$0 < c_5 \leq .7, \quad .7 < c_6 \leq .9$$

Because the requirements are discrete random variables, considerable variation of the risk coefficients c_j is allowed without changing the requirements in the deterministic equivalent. Nevertheless, the optimal

solution is sensitive to changes in c_j if c_j is near the extremes of the allowable range.

To sum up, the model presented in this section determines an optimal (minimum expected time) distribution of units from their original locations to contact areas. The number of units which will be required to accomplish the mission in each area is not known in advance, but these requirements are distributed according to a known c. d. f. We required that the number of units sent to each contact area must satisfy the actual requirement with at least a specified probability, i. e., the mission must be accomplished with at least this specified probability. The model was then reduced to an equivalent distribution model with no random elements (a deterministic equivalent). Finally, an example was given to illustrate these ideas.

5.5 A Dynamic Distribution Model

We extend the model of the previous section to two periods. This extension is dynamic because the two periods are coupled together by using the same units in each period. The problem is to minimize the expected distribution time, subject to constraints on the amounts available and chance constraints on the requirements. This model is reduced to a deterministic equivalent by the use of a zero-order decision rule for each period. A method is also proposed which allows the decision variables in the second period to be dependent on the actual requirements observed in the first period. The method only utilizes zero-order decision rules.

To formulate the model, we assume that allocations are made at the beginning of periods 1 and 2. We number the locations where units are available and/or required $j = 1, \dots, n$. Because of this numbering system, some of these locations may have either nothing available or nothing required. Let d_j^k be the number of units required to accomplish the specified mission at location j in period $k = 1, 2$. Now d_j^1 is observed after the first and before the second allocations are made, while d_j^2 is observed after the second allocations are made. The d_j^k are discrete random variables with a known joint c. d. f. Let x_{ij}^k be the number of units sent from location i to j ($i, j = 1, \dots, n$) in period $k = 1, 2$, and let a_i be the number of units available at location i

initially. We introduce the following distribution constraints:

$$(27) \quad \sum_j x_{ij} \leq a_i$$

$$(28) \quad \Pr \left\{ \sum_i x_{ij} \geq d_j^1 \right\} \geq c_j^1$$

$$(29) \quad \sum_j x_{ij}^2 - \sum_j x_{ji}^1 + \sum_j x_{ij}^1 \leq a_i$$

$$(30) \quad \Pr \left\{ \sum x_{ij}^2 \geq d_j^2 \right\} \geq c_j^2$$

$$(31) \quad x_{ij}^1, x_{ij}^2 \geq 0$$

where c_j^1 and c_j^2 are specified constants with $0 \leq c_j^k \leq 1$.

Constraint (29) is the only type not encountered in the last section. It requires that the amount sent from location i in period two

$\left(\sum_j x_{ij}^2 \right)$ cannot exceed the amount available there

$\left(a_i + \sum_j x_{ji}^1 - \sum_j x_{ij}^1 \right)$. This constraint couples the distribution models of each period.

Let t_{ij}^k be the time taken by one unit to go from location i to j in period k (t_{ij}^k may be a random variable). Our objective is to minimize the total expected distribution time, i. e.,

$$(32) \quad \text{Min } \sum_{i,j} \hat{t}_{ij}^1 x_{ij}^1 + \hat{t}_{ij}^2 x_{ij}^2$$

where \hat{t}_{ij}^1 and \hat{t}_{ij}^2 denote the means of t_{ij}^1 and t_{ij}^2 respectively.

We assume that both x_{ij}^1 and x_{ij}^2 are determined by zero-order decision rules; then, by Lemma 1 of the last section, equations (28)

and (10) are equivalent to

$$(33) \quad \sum_i x_{ij}^k \geq \bar{F}_j^k(c_j^k) \quad k = 1, 2$$

where $\bar{F}_j^k(c_j^k)$ is an integer determined from the marginal c. d. f. of d_j^k (see equation (25)).

From the above discussion, the model with objective (32) and constraints (27) through (31) is equivalent to the following deterministic model

$$(34) \quad \begin{aligned} \text{Min } & \sum_{i,j} \hat{t}_{ij}^1 x_{ij}^1 + \hat{t}_{ij}^2 x_{ij}^2 \\ & \sum_j x_{ij}^1 \leq a_i \\ & \sum_i x_{ij}^1 \geq \bar{F}_j^1(c_j^1) \\ & \sum_j x_{ij}^1 - \sum_j x_{ji}^1 + \sum_j x_{ij}^2 \leq a_i \\ & \sum_i x_{ij}^2 \geq \bar{F}_j^2(c_j^2) \\ & x_{ij}^1, x_{ij}^2 \geq 0 \end{aligned}$$

Now (34) has an optimal solution if and only if

$$\sum_i a_i \geq \sum_{j=1}^n \bar{F}_j^k(c_j^k) \quad k = 1, 2^1$$

In addition, (34) has special structural features. Indeed, techniques

¹This follows immediately from the distribution properties of (34).

such as the mixing routine of Charnes and Cooper [5] are available to exploit this special structure and thereby reduce computational effort. However, for multiple contact situations, "n" would be on the order of 10. In this case, (34) is a linear program with 200 variables and 40 constraints. This is not a large problem for modern linear program codes. Therefore, we do not dwell on special methods of computations.

We propose a method for implementing (34). An optimal solution to (34) yields optimal x_{ij}^1 and x_{ij}^2 . Instead of using both of these optimal distribution plans, one can employ the following procedure. Use the optimal x_{ij}^1 from (34). The x_{ij}^2 which are actually employed are obtained by solving the one period distribution model of section 5.4, with F_j^2 being the conditional distribution of d_j^2 given the actual values of d_j^1 which have been observed. To reiterate, the optimal x_{ij}^1 from (34) are used. The x_{ij}^2 from (34) are not used. Rather, we determine conditionally optimal x_{ij}^2 given the actual values of d_j^1 which are observed.

This dynamic model can also be formulated with a linear decision rule. Only a verbal description of the procedure is given here, since no new results are obtained when such a rule is employed. We would write x_{ij}^2 as an unknown linear combination of the random variables d_j^1 . These expressions for x_{ij}^2 are then substituted in (34), and a deterministic equivalent for the case of normally distributed d_j^1

and d_j^2 can be obtained by the method of Charnes and Cooper [4]. The optimal solution to the resulting model yields an optimal linear decision rule for the x_{ij}^2 . The x_{ij}^2 which are employed will, therefore, depend on the actual values of d_j^1 which are observed. Hence, an adaptive model is obtained by the above procedure.

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VITA

The author, Roger Glenn Schroeder, [REDACTED]

[REDACTED] He received a Bachelor of Science degree with high distinction in 1962 and a Master of Science degree in Industrial Engineering in 1963. Both of these degrees are from the University of Minnesota.

His publications are:

1. Schroeder, R. G., and G. J. Rath, "The Role of Mathematical Models in Educational Research", Psychology in the Schools, Vol. 2, No. 4, October, 1965, pp. 317-325.
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